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CONSTRUCTION OF WEIGHT TWO EIGENFORMS VIA THE GENERALIZED DEDEKIND ETA FUNCTION

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ABSTRACT. The generalized Dedekind eta function has been used in various ways to construct modular functions of different weights. In this paper we give a way to construct modular forms of weight two for the modular groups $\Gamma_0(N)$ which, in some cases, turn out to be Hecke eigenforms (though never cusp forms).

1. The generalized Dedekind eta function. Let $\mathfrak{h}$ denote the upper half plane (so $\mathfrak{h} = \{\tau \mid \operatorname{Im}\tau > 0\}$), and let $P_2(x) = \{x\}^2 - \{x\} + (1/6)$ denote the second Bernoulli polynomial, defined on the fractional part of x, $\{x\} = x - \lfloor x \rfloor$. For integers g and δ, with $\delta > 0$, we define the generalized Dedekind eta function as

$$(1) \quad \eta_{\delta,g}(\tau) = e^{\pi i \delta P_2(g/\delta)\tau} \prod_{\substack{m \equiv g \pmod{\delta} \\ m>0}} (1-q^m) \prod_{\substack{m \equiv -g \pmod{\delta} \\ m>0}} (1-q^m)$$

where $\tau \in \mathfrak{h}$ and $q = e^{2\pi i \tau}$. These functions are a variation of the eta functions defined by Schoeneberg in [**5**] and can be used to create modular functions in various ways (see [**4**] and [**6**]). For example, from [**6**], we have

Theorem. *Let N be a positive integer, and let*

$$f(\tau) = \prod_{\substack{\delta \mid N \\ 0 \le g < \delta}} \eta_{\delta,g}^{r_{\delta,g}}(\tau),$$

where $r_{\delta,g} \in Z$ and $r_{\delta,ag} = r_{\delta,g}$ for all a relatively prime to N. Set

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$k = \sum_{\delta|N} r_{\delta,0}$. If

$$\sum_{\substack{\delta|N\\0\le g<\delta}} \delta P_2\left(\frac{g}{\delta}\right) r_{\delta,g} \equiv 0 \pmod 2,$$

$$\sum_{\substack{\delta|N\\0\le g<\delta}} \frac{N}{6\delta}\, r_{\delta,g} \equiv 0 \pmod 2,$$

and if k is an even integer, then f is a modular function of weight k on $\Gamma_0(N)$.

2. The functions $H_{\delta,g}(\tau)$. In this paper we will consider a class of modular forms, reminiscent of the Eisenstein series, of weight two, derived from the generalized Dedekind eta function.

First recall that, for $A = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \Gamma_0(\delta)$ and $g \not\equiv 0 \pmod{\delta}$,

$$\eta_{\delta,g}(A\tau) = \nu_{\delta,g}(A)\eta_{\delta,ag}(\tau)$$

where

$$\nu_{\delta,g}(A) = \begin{cases} \exp\left(\pi i\left[\frac{a}{c}\,\delta P_2(g/\delta) + \frac{d}{c}\,\delta P_2\left(\frac{ag}{\delta}\right)\right.\right. & \\ \qquad\qquad \left.\left. -2\operatorname{sgn} c \cdot s(a, c/\delta; 0, g/\delta)\right]\right) & \text{if } c \neq 0 \\ \exp\left(\pi i \frac{b}{d}\,\delta P_2(g/\delta)\right) & \text{if } c = 0, \end{cases}$$

and $s(h,k;x,y)$ is the generalized Dedekind sum (see [4] and [**5**]). Let

$$H_{\delta,g}(\tau) = \frac{1}{2\pi i}\,\frac{\eta'_{\delta,g}(\tau)}{\eta_{\delta,g}(\tau)}. \tag{2}$$

Since $\eta_{\delta,g}(\tau)$ is holomorphic and nonzero on $\mathfrak{h}$, $H_{\delta,g}(\tau)$ is holomorphic on $\mathfrak{h}$. We now consider what happens at the cusps. The function $\eta_{\delta,g}(\tau)$ is meromorphic at any cusp γ of $\mathfrak{h}$ (see [**5**]), so

$$\eta_{\delta,g}(A\tau) = \sum_{n=M}^{\infty} a_n q_\delta^n,$$

where $M \in \mathbf{Z}$ and $q_\delta = e^{2\pi i\tau/\delta}$. Differentiating both sides of the above equation with respect to τ yields

$$(c\tau + d)^{-2}\, \eta'_{\delta,g}(A\tau) = \sum_{n=M}^{\infty} a_n \left(\frac{2\pi i}{\delta}\right) n q_\delta^n;$$

in particular, if $\eta_{\delta,g}(\tau)$ has a pole of order M at γ, then $\eta'_{\delta,g}(\tau)$ also has a pole of order M at γ. Similarly, if $\eta_{\delta,g}(\tau)$ has a zero of order M at γ, then $\eta'_{\delta,g}(\tau)$ also has a zero of order M at γ. Consequently, $H_{\delta,g}(\tau)$ will be holomorphic at γ.

Of special interest is the expansion of $H_{\delta,g}(\tau)$ at infinity. We find this by looking at the logarithmic derivative of the expansion of $\eta_{\delta,g}(\tau)$ at infinity: starting with (1) we have

(3)

$$\begin{aligned}
&\log\eta_{\delta,g}(\tau)\\
&\quad= \pi i\delta P_2(g/\delta)\tau + \sum_{\substack{m\equiv g \pmod{\delta}\\ m>0}} \log(1-q^m) + \sum_{\substack{m\equiv -g \pmod{\delta}\\ m>0}} \log(1-q^m)\\
&\quad= \pi i\delta P_2(g/\delta)\tau - \sum_{\substack{m\equiv g \pmod{\delta}\\ m>0}} \sum_{n=1}^{\infty} \frac{q^{mn}}{n} - \sum_{\substack{m\equiv -g \pmod{\delta}\\ m>0}} \sum_{n=1}^{\infty} \frac{q^{mn}}{n}.
\end{aligned}$$

Differentiating (3) with respect to τ yields

$$\begin{aligned}
\frac{\eta'_{\delta,g}}{\eta_{\delta,g}}(\tau) &= \pi i\delta P_2\left(\frac{g}{\delta}\right) - \sum_{\substack{m\equiv g \pmod{\delta}\\ m>0}} \sum_{n=1}^{\infty} 2\pi i m q^{mn}\\
&\qquad - \sum_{\substack{m\equiv -g \pmod{\delta}\\ m>0}} \sum_{n=1}^{\infty} 2\pi i m q^{mn}\\
&= \pi i\delta P_2\left(\frac{g}{\delta}\right) - 2\pi i \sum_{n=1}^{\infty} \left(\sum_{\substack{m\equiv g \pmod{\delta}\\ m>0}} m + \sum_{\substack{m\equiv -g \pmod{\delta}\\ m>0}} m \right) q^{mn}\\
&= \pi i\delta P_2\left(\frac{g}{\delta}\right) - 2\pi i \sum_{N=1}^{\infty} \left(\sum_{\substack{m\equiv g \pmod{\delta}\\ m>0}} m + \sum_{\substack{m\equiv -g \pmod{\delta}\\ m>n}} m \right) q^{N}.
\end{aligned}$$

Let

$$\sigma_{\delta,g}(N) = \sum_{\substack{d|N \\ d\equiv g \pmod{\delta}}} d + \sum_{\substack{d|N \\ d\equiv -g \pmod{\delta}}} d;$$

then the expansion of $H_{\delta,g}(\tau)$ at infinity can be written as

$$H_{\delta,g}(\tau) = \frac{1}{2\pi i}\,\frac{\eta'_{\delta,g}}{\eta_{\delta,g}}(\tau) = \frac{1}{2}\,\delta P_2(g/\delta) - \sum_{N=1}^{\infty} \sigma_{\delta,g}(N) q^N. \tag{4}$$

The coefficients of this expansion can be used to derive combinatorial results. For example, the fourth power of the classical theta function $\bigl(\theta(\tau) = \sum_{n\in\mathbf{Z}} q^{n^2}\bigr)$ can be written as $1+\sum_{N=1} s_4(N)q^N$, where $s_4(N)$ denotes the number of ways of writing N as a sum of four squares. One can show that $\theta^4(\tau) = (1/3)H_{4,1}(\tau) + (2/3)H_{4,2}(\tau)$, which gives the formula for $s_4(N)$:

$$s_4(N) = 8\sigma_{4,1}(N) + 4\sigma_{4,2}(N).$$

Similarly, one can find a formula for the number of ways a positive integer N can be written as a sum of four triangular numbers:

$$t_4(N) = \sigma_{4,1}(2N+1) = \sigma(2N+1),$$

where $t_4(N)$ denotes the number of ways of writing N as a sum of four triangular numbers (see [**6**] for the details of these derivations).

We can use the transformation formula of $\eta_{\delta,g}(\tau)$ to find a transformation formula for $H_{\delta,g}(\tau)$: if $A \in \Gamma_0(\delta)$, then

$$\eta_{\delta,g}(A\tau) = \nu_{\delta,g}(A)\eta_{\delta,ag}(\tau);$$

this implies (after differentiating by τ) that

$$\eta'_{\delta,g}(A\tau) = \nu_{\delta,g}(A)(c\tau+d)^2\eta'_{\delta,ag}(\tau).$$

Therefore,

$$\begin{aligned} H_{\delta,g}(A\tau) &= \frac{1}{2\pi i}\,\frac{\eta'_{\delta,g}(A\tau)}{\eta_{\delta,g}(\tau)} \\ &= (c\tau+d)^2\,\frac{1}{2\pi i}\,\frac{\eta'_{\delta,ag}}{\eta_{\delta,ag}}\,(\tau) \\ &= (c\tau+d)^2\,H_{\delta,ag}(\tau). \end{aligned}$$

Suppose that $(\delta/g) = 2, 3, 4$ or 6. If $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(\delta)$, then $(a, \delta) = 1$ and hence $ag \equiv \pm g \pmod{\delta}$. So

$$H_{\delta,g}(A\tau) = (c\tau + d)^2\, H_{\delta,ag}(\tau) = (c\tau + d)^2\, H_{\delta,g}(\tau),$$

which implies that $H_{\delta,g}(\tau)$ is a modular form of weight two on $\Gamma_0(\delta)$.

As an example, consider $H_{2,1}(\tau)$. This is a modular form of weight two on $\Gamma_0(2)$. Using the formulas of Shimura and Gunning (see [**6**]), we find that the space of modular forms of weight two on $\Gamma_0(2)$ has dimension one, so that, in fact, $H_{2,1}(\tau)$ is the modular form of weight two on $\Gamma_0(2)$. In particular, it is the eigenform (with respect to the Hecke transform) for $\Gamma_0(2)$. We will discuss eigenforms more in Section 4.

When (δ/g) is not 2, 3, 4, or 6, then the function $H_{\delta,g}(\tau)$ may not be a modular form on $\Gamma_0(\delta)$. For example, $H_{5,1}(\tau)$ is not a modular form for $\Gamma_0(5)$:

$$H_{5,1}\left(\frac{2\tau + 1}{5\tau + 3}\right) = (5\tau + 3)^2\, H_{5,2}(\tau) \neq (5\tau + 3)^2\, H_{5,1}(\tau).$$

However, note that for $A \in \Gamma_1(\delta)$, we always have $H_{\delta,g}(A\tau) = (c\tau + d)^2 H_{\delta,g}(\tau)$, since $a \equiv 1 \pmod{\delta}$; hence, $H_{\delta,g}(\tau)$ is always a modular form of weight two on $\Gamma_1(\delta)$.

3. Constructing modular forms. We now focus exclusively on $\Gamma_0(\delta)$. Let $M_k(\Gamma')$ denote the vector space of modular forms of weight k on Γ'. Based on the previous section, we have results such as

$$\begin{aligned} H_{2,1}(\tau) &= \frac{1}{2}\,(2)P_2\left(\frac{1}{2}\right) - \sum_{N=1}^{\infty} \sigma_{2,1}(N)q^N \\ &= -\frac{1}{12} - \sum_{N=1}^{\infty} \sigma_{2,1}(N)q^N \in M_2(\Gamma_0(2)), \\ H_{6,1}(\tau) &= \frac{1}{2}\,(6)P_2\left(\frac{1}{6}\right) - \sum_{N=1}^{\infty} \sigma_{6,1}(N)q^N \\ &= \frac{1}{12} - \sum_{N=1}^{\infty} \sigma_{6,1}(N)q^N \in M_2(\Gamma_0(6)), \end{aligned}$$

and

$$\begin{aligned} H_{20,5}(\tau) &= \frac{1}{2}\,(20)P_2\left(\frac{5}{20}\right) - \sum_{N=1}^{\infty} \sigma_{20,5}(N)q^N \\ &= -\frac{5}{24} - \sum_{N=1}^{\infty} \sigma_{20,5}(N)q^N \in M_2(\Gamma_0(20)). \end{aligned}$$

Although $H_{5,1}(\tau)$ and $H_{5,2}(\tau)$ are not modular on $\Gamma_0(5)$, they can be used to construct modular forms for $\Gamma_0(5)$, in some cases with a character as a multiplier. For example, let $F(\tau) = H_{5,1}(\tau) + H_{5,2}(\tau) + H_{5,3}(\tau) + H_{5,4}(\tau)$. Then, for $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(5)$, we have

$$\begin{aligned} F(A\tau) &= H_{5,1}(A\tau) + H_{5,2}(A\tau) + H_{5,3}(A\tau) + H_{5,4}(A\tau) \\ &= (c\tau+d)^2\big(H_{5,a}(\tau) + H_{5,2a}(\tau) + H_{5,3a}(\tau) + H_{5,4a}(\tau)\big) \\ &= (c\tau+d)^2 F(\tau), \end{aligned}$$

and thus $F(\tau) \in M_2(\Gamma_0(5))$. Similarly, suppose χ is the quadratic character defined by the Legendre symbol modulo 5: $\chi(a) = (a/5)$. Now let $G(\tau) = H_{5,1}(\tau) - H_{5,2}(\tau) - H_{5,3}(\tau) + H_{5,4}(\tau)$. Then

$$\begin{aligned} G(A\tau) &= H_{5,1}(A\tau) - H_{5,2}(A\tau) - H_{5,3}(A\tau) + H_{5,4}(A\tau) \\ &= (c\tau+d)^2\big(H_{5,a}(\tau) - H_{5,2a}(\tau) - H_{5,3a}(\tau) + H_{5,4a}(\tau)\big). \end{aligned}$$

If $a \equiv \pm 1 \pmod 5$, then $G(A\tau) = (c\tau+d)^2 G(\tau)$. If $a \equiv \pm 2 \pmod 5$, then $G(A\tau) = (c\tau+d)^2\big(H_{5,2}(\tau) - H_{5,4}(\tau) - H_{5,1}(\tau) + H_{5,3}(\tau)\big) = -(c\tau+d)^2 G(\tau)$. We summarize this by writing

$$G(A\tau) = \left(\frac{a}{5}\right)(c\tau+d)^2\, G(\tau).$$

Since $(a/5) = (d/5)$ for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(5)$, we have shown that $G(A\tau) = \chi(d)(c\tau+d)^2 G(\tau)$, and hence G is a modular form of weight two on $\Gamma_0(5)$ with χ as a multiplier.

We generalize the last two examples with the following.

Theorem 1. *Let χ be a Dirichlet character modulo N (N a positive integer), and set*

$$f(\tau) = \sum_{k=1}^{N} \chi(k)\, H_{N,k}(\tau).$$

Then f is a modular form of weight two on $\Gamma_0(N)$ with multiplier χ.

Proof. Each $H_{N,k}(\tau)$ is holomorphic at the cusps of $\Gamma_0(N)$, so we need only check the transformation formula. Taking $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$, we find

$$\begin{aligned} f(A\tau) &= \sum_{k=1}^{N} \chi(k)\, H_{N,k}(A\tau) \\ &= \sum_{k=1}^{N} \chi(k)(c\tau+d)^2\, H_{N,ak}(\tau) \\ &= \bar{\chi}(a) \sum_{k=1}^{N} \chi(ak)(c\tau+d)^2\, H_{N,ak}(\tau) \\ &= \bar{\chi}(a)(c\tau+d)^2 \sum_{k=1}^{N} \chi(ak)\, H_{N,ak}(\tau) \\ &= \bar{\chi}(a)(c\tau+d)^2 f(\tau). \end{aligned}$$

Since $ad \equiv 1 \pmod N$, we can replace $\bar{\chi}(a)$ with $\chi(d)$. This gives

$$f(A\tau) = \chi(d)(c\tau+d)^2 f(\tau),$$

which implies that f is a modular form of weight 2 on $\Gamma_0(N)$ with multiplier χ. $\square$

Note that in the case where $\chi(-1) = -1$, the function f is the zero function. If $N = 5$ and $\chi(n)$ is the trivial Dirichlet character modulo 5, then $f(\tau)$ is the function $F(\tau)$ defined above. Similarly, if $N = 5$ and $\chi(n) = (n/5)$, then $f(\tau) = G(\tau)$.

4. Eigenforms. According to Theorem 1, the functions $F(\tau)$ and $G(\tau)$ are modular forms of weight two with the corresponding characters as multipliers. In fact, they are eigenforms with respect to the Hecke operator. For example, consider $F(\tau)$. For any positive integer m, let $\lambda_m = \sum_{d|m, 5\nmid d} d = \sum_{d|m} \chi(d)d$, where χ denotes the trivial character modulo 5. If T_m denotes the Hecke operator, then $T_m(F) = \lambda_m F$. To show this, we need the following.

Lemma. *Let χ be a Dirichlet character. Then, for any positive integers m and n,*

$$\Big(\sum_{d|m}\chi(d)d\Big)\Big(\sum_{d|n}\chi(d)d\Big)=\sum_{d|\gcd(m,n)}\Big(\chi(d)d\sum_{e|(mn/d^2)}\chi(e)e\Big).$$

Proof. According to Theorem 1.12 of [**3**], the following are equivalent:

(1) The arithmetic function g is the convolution of two completely multiplicative functions.

(2) There is a completely multiplicative function B such that for all positive integers m and n,

$$g(m)g(n)=\sum_{d|\gcd(m,n)}B(d)g\Big(\frac{mn}{d^2}\Big);$$

in particular, the function B is determined by $B(p)=g(p)^2-g(p^2)$ for any prime p.

We apply this result to the arithmetic function $g(n)=\sum_{d|n}\chi(d)d$. Since g is the convolution of $x(n)n$ and 1, both of which are completely multiplicative, we can write

$$\Big(\sum_{d|m}\chi(d)d\Big)\Big(\sum_{d|n}\chi(d)d\Big)=\sum_{d|\gcd(m,n)}B(d)\Big(\sum_{e|(mn/d^2)}\chi(e)e\Big),$$

where B is some completely multiplicative function. In particular, B is determined by the relation $B(p)=g(p)^2-g(p^2)=(1+\chi(p)p)^2-(1+\chi(p)p+\chi(p^2)p^2)=\chi(p)p$, which gives the desired result. □

Since

$$\begin{aligned}F(\tau)&=H_{5,1}(\tau)+H_{5,2}(\tau)+H_{5,3}(\tau)+H_{5,4}(\tau)\\&=-\frac{1}{3}-\sum_{N=1}^{\infty}\big(\sigma_{5,1}(N)+\sigma_{5,2}(N)+\sigma_{5,3}(N)+\sigma_{5,4}(N)\big)q^N,\end{aligned}$$

we can write the Fourier expansion of $F(\tau)$ as $\sum a_nq^n$ where $a_0=-1/3$ and $a_n=\sigma_{5,1}(n)+\sigma_{5,2}(n)+\sigma_{5,3}(n)+\sigma_{5,4}(n)=\sum_{d|n}\chi(d)d$. Then

$T_m F = \sum b_n q^n$ where

$$b_n = \sum_{d \mid \gcd(m,n)} \chi(d)\, d a_{mn/d^2}$$

(see [**1**, Proposition 39]). Now

$$\lambda_m a_n = \Big(\sum_{d \mid m} \chi(d) d\Big)\Big(\sum_{d \mid n} \chi(d) d\Big),$$

and

$$\begin{aligned} b_n &= \sum_{d \mid \gcd(m,n)} \chi(d)\, d a_{mn/d^2} \\ &= \sum_{d \mid \gcd(m,n)} \Big(\chi(d) d \sum_{e \mid (mn/d^2)} \chi(e) e\Big). \end{aligned}$$

By the lemma, $\lambda_m a_n = b_n$, and thus $T_m F = \lambda_m F$. A similar argument shows that $G(\tau)$ is an eigenform with respect to the quadratic character $(\cdot/d)$.

The function defined in Theorem 1 is always a modular form of weight two on $\Gamma_0(N)$ with χ as a multiplier. We conclude by showing that, if the function is not the zero function, then it turns out to be an eigenform:

Theorem 2. *Let χ be a Dirichlet character modulo N (N a positive integer), with $\chi(-1) = 1$, and set*

$$f(\tau) = \sum_{k=1}^{N} \chi(k)\, H_{N,k}(\tau).$$

Then f is an eigenform of weight two on $\Gamma_0(N)$ with multiplier χ.

Proof. Let m be a positive integer, and write $f(\tau)$ as $\sum a_n q^n$ and $T_m f$ as $\sum b_n q^n$. Then for $n > 0$, $a_n = \sum_{d \mid n} \chi(d) d$. Let $\lambda_m = \sum_{d \mid m} \chi(d) d$. Then

$$\lambda_m a_n = \Big(\sum_{d \mid m} \chi(d) d\Big)\Big(\sum_{d \mid n} \chi(d) d\Big),$$

and

$$\begin{aligned} b_n &= \sum_{d \mid \gcd(m,n)} \chi(d)\, d a_{mn/d^2} \\ &= \sum_{d \mid \gcd(m,n)} \left(\chi(d) d \sum_{e \mid (mn/d^2)} \chi(e) e \right). \end{aligned}$$

By the lemma, $\lambda_m a_n = b_n$, and hence f is an eigenform. □

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