V. Other Possible Applications

Having used the generalized Dedekind eta functions to study sums of triangular, square, pentagonal, and octagonal numbers, we could try the same approach with other figurate numbers (or with any quadratic form $an^2 + bn$ where $a > |b|$). For example, if we wanted to look at septagonal numbers, we let $a = 5/2$ and $b = 3/2$ in Theorem 2 and look at the function

$$F(\tau) = q^{9/40} \sum_{n \in \mathbb{Z}} q^{\frac{5}{2}n^2 - \frac{3}{2}n} = \eta_{5,0}\, \eta_{10,2}\, \eta_{5,1}(\tau).$$

Note that Theorem 1 can not be applied since $r_{5,1} \neq r_{5,2}$. However, we could use [21] to determine if $F^k(\tau)$ is modular on $\Gamma_1(N)$ for some k and N. Note that

$$\sum_{\substack{\delta | N \\ 0 \le g < \delta}} r_{\delta,g} \delta P_2\left(\frac{g}{\delta}\right) = \frac{1}{2} \cdot 5 \cdot \frac{1}{6} + 10 \cdot \frac{1}{150} - 5 \cdot \frac{1}{150} = \frac{9}{20}$$

and

$$\sum_{\substack{\delta | N \\ 0 \le g < \delta}} r_{\delta,g} \frac{N}{6\delta} = \frac{N}{6}\left(\frac{1}{2} \cdot \frac{1}{5} + \frac{1}{10} - \frac{1}{5}\right) = 0.$$

Thus, we could examine sums of two general rank septagonal numbers by looking at the function

$$F^2(20\tau) = \eta_{100,0}\, \eta_{200,40}^2\, \eta_{100,20}^{-2}(\tau) = \sum_{N=0}^{\infty} a(N) q^{20N+9},$$

(which is a modular form of weight 1 on $\Gamma_1(200)$) where $a(N)$ denotes the number of ways of writing N as a sum of two general rank septagonal numbers. Similarly, we could consider sums of four septagonal numbers by looking at

$$F^4(10\tau) = \eta_{50,0}^2\, \eta_{100,20}^4\, \eta_{50,10}^{-4}(\tau) = \sum_{N=0}^{\infty} b(N) q^{10N+9},$$

(which is a modular form of weight 2 on $\Gamma_1(100)$) where $b(N)$ denotes the number of ways of writing N as a sum of four general rank septagonal numbers. There is one daunting disadvantage, though: the dimension of $\Gamma_1(N)$ tends to be much higher than the dimension of $\Gamma_0(N)$ ([13],Proposition 8 of chapter 3).

The Rogers-Ramanujan identities,

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{m=0}^{\infty}(1-q^{5m+1})^{-1}(1-q^{5m+4})^{-1}$$

and

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{m=0}^{\infty}(1-q^{5m+2})^{-1}(1-q^{5m+3})^{-1}$$

can be written in terms of generalized Dedekind eta functions:

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} = q^{1/60}\eta_{5,1}^{-1}(\tau)$$

and

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)} = q^{-11/60}\eta_{5,2}^{-1}(\tau).$$

Also, Gordon's generalization of these identities (Theorem 7.8 of [1]):

Let $k \geq 2$ and 1 □ i □ k. Then

$$\sum_{n_1,\ldots,n_{k-1}\geq 0} \frac{q^{N_1^2+\cdots+N_{k-1}^2+N_i+\cdots+N_{k-1}}}{(q)_{n_1}\cdots(q)_{n_{k-1}}} = \prod_{\substack{n=1 \\ n\not\equiv 0,\pm i \ (\mathrm{mod}\ 2k+1)}}^{\infty} (1-q^n)^{-1}$$

where $N_j = n_j + \cdots + n_{k-1}$ and $(q)_m = (1-q)\cdots(1-q^m)$,

can be written as

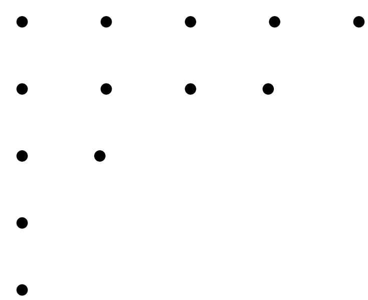
$$\sum_{n_1,\ldots,n_{k-1}\geq 0} \frac{q^{N_1^2+\cdots+N_{k-1}^2+N_i+\cdots+N_{k-1}}}{(q)_{n_1}\cdots(q)_{n_{k-1}}} = q^{-r}\eta_{1,0}^{-1/2}\,\eta_{2k+1,0}^{1/2}\,\eta_{2k+1,i}(\tau),$$

where $r = \frac{k}{12} + (k+\frac{1}{2})P_2\left(\frac{i}{2k+1}\right)$.

The generalized Dedekind eta functions are also connected to certain specialized partitions. Since many of the generating functions for partitions involve $q-$products, they can be studied using eta functions and modular forms. The classical example is $p(n)$, which denotes the number of ordinary partitions of n. Its generating function is $\Phi(q) = \prod_{m=1}^{\infty}(1-q^m)^{-1}$; in other words, $\Phi(q) = q^{-1/24}\eta^{-1}(\tau) = q^{-1/24}\eta_{1,0}^{-1/2}(\tau)$.

The transformation formula for the eta function can be used to prove that $p(n)$, using the circle method of Hardy and Littlewood.

One specialized partition which can be studied using modular forms, is the $t-$core partition. Given a partition $n = \lambda_1 + \lambda_2 + \cdots + \lambda_r$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$, let λ'_i denote the number of nodes in the ith column of the Ferrers-Young diagram of the partition. For the node (i, j) (the node in the ith row and jth column), let $H(i, j)$ denote the hook number, which is $\lambda_i + \lambda'_j - i - j + 1$. A $t-$core partition is a partition of n in which none of the hook numbers are multiples of t. For example, for the partition $13 = 5+4+2+1+1$, we have the following Ferrers-Young diagram:



and the following values:

$$\lambda_1 = 5 \qquad \lambda_2 = 4 \qquad \lambda_3 = 2 \qquad \lambda_4 = \lambda_5 = 1$$

$$\lambda'_1 = 5 \qquad \lambda'_2 = 3 \qquad \lambda'_3 = \lambda'_4 = 2 \qquad \lambda'_5 = 1$$

$$H(1,1) = 5+5-1-1+1 = 9 \qquad H(2,3) = 4+2-2-3+1 = 2$$

$$H(1,2) = 5+3-1-2+1 = 6 \qquad H(2,4) = 4+2-2-4+1 = 1$$

$$H(1,3) = 5+2-1-3+1 = 4 \qquad H(3,1) = 2+5-3-1+1 = 4$$

$$H(1,4) = 5+2-1-4+1 = 3 \qquad H(3,2) = 2+3-3-2+1 = 1$$

$$H(1,5) = 5+1-1-5+1 = 1 \qquad H(4,1) = 1+5-4-1+1 = 2$$

$$H(2,1) = 4+5-2-1+1 = 7 \qquad H(5,1) = 1+5-5-1+1 = 1$$

$$H(2,2) = 4+3-2-2+1 = 4.$$

Thus, this partition is a $t-$core partition if $t = 5, t = 8$, or $t > 9$.

The notion of $t-$core partitions has applications in the representations of symmetric groups [11] and the theory of cranks [6]. If $c_t(n)$ denotes the number of $t-$core partitions of n, then we have the following relation:

$$\sum_{n=0}^{\infty} c_t(n)q^n = \prod_{m=1}^{\infty} \frac{(1-q^{tm})^t}{1-q^m}$$

see [6] or [17]. For $t = 2$, we have

$$\begin{aligned}\sum_{n=0}^{\infty} c_2(n)q^n &= \prod_{m=1}^{\infty} \frac{(1-q^{2m})^2}{1-q^m} \\ &= 1 + q + q^3 + q^{10} + \dots\end{aligned}$$

and for $t = 3$ we have

$$\begin{aligned}\sum_{n=0}^{\infty} c_3(n)q^n &= \prod_{m=1}^{\infty} \frac{(1-q^{3m})^3}{1-q^m} \\ &= 1 + q + 2q^2 + 2q^4 + q^5 + 2q^6 + \dots.\end{aligned}$$

If t is a prime greater than 5, then

$$\begin{aligned}\sum_{n=0}^{\infty} c_t(n)q^n &= \prod_{m=1}^{\infty} \frac{(1-q^{tm})^t}{1-q^m} \\ &= q^{-(t^2-1)/24}\eta_{t,0}^{t/2}\,\eta_{1,0}^{-1/2}(\tau).\end{aligned}$$

For the function $\eta_{t,0}^{t/2}\,\eta_{1,0}^{-1/2}(\tau)$, the congruences of Theorem 1 yield

$$\sum_{\substack{\delta|N \\ 0\le g<\delta}} r_{\delta,g}\delta P_2\Big(\frac{g}{\delta}\Big) = \frac{t}{2}\cdot t\cdot\frac{1}{6} - \frac{1}{2}\cdot\frac{1}{6} = \frac{t^2-1}{12} \equiv 0 \pmod 2$$

and

$$\sum_{\substack{\delta|N \\ 0\le g<\delta}} r_{\delta,g}\frac{N}{\delta}\frac{1}{6} = \frac{N}{6}\Big(\frac{t}{2}\cdot\frac{1}{t} - \frac{1}{2}\Big) = 0.$$

Since t is prime and greater than 5, $\frac{t^2-1}{12}$ will be an even integer. So the function $\eta_{t,0}^{t/2}\,\eta_{1,0}^{-1/2}(\tau)$ has order $\frac{t^2-1}{24}$ at the cusp at infinity and order 0 at the cusp 0. These are the only cusps of $\Gamma_0(t)$, and hence $\eta_{t,0}^{t/2}\,\eta_{1,0}^{-1/2}(\tau)$ is a modular form of weight $\frac{t-1}{2}$ on $\Gamma_0(t)$ with multiplier $i^{(a-1)(t-1)/2}\Big(\frac{a}{t}\Big)$.

A further specialization of these $t-$core partitions is the notion of self-conjugate $t-$core partitions. Recall that the partition $n = \lambda_1 + \lambda_2 + \cdots + \lambda_r$ is self-conjugate is $\lambda_i = \lambda_i'$ for $i = 1, 2, \ldots, r$. The generating function for self-conjugate $t-$core partitions is

$$\sum_{n=0}^{\infty} \mathrm{asc}_t(n)q^n = q^{-(t^2-1)/24} \frac{\eta_{4,2}^{1/2}}{\eta_{2,1}^{1/2}} \frac{\eta_{2t,t}^{1/2}}{\eta_{4t,2t}^{1/2}} \eta_{2t,0}^{(t-1)/4}(\tau)$$

if t is odd and

$$\sum_{n=0}^{\infty} \mathrm{asc}_t(n)q^n = q^{-(t^2-1)/24} \frac{\eta_{4,2}^{1/2}}{\eta_{2,1}^{1/2}} \eta_{2t,0}^{t/4}(\tau)$$

if t is even ((7.1) of [6]). Let $F_t(\tau) = q^{(t^2-1)/24} \sum_{n=0}^{\infty} \mathrm{asc}_t(n)q^n$. Then

$$\sum_{\substack{\delta | N \\ 0 \square\, g<\delta}} r_{\delta,g}\delta P_2\left(\frac{g}{\delta}\right) = \frac{t^2-1}{12} \quad \text{and} \quad \sum_{\substack{\delta | N \\ 0 \square\, g<\delta}} r_{\delta,g}\frac{N}{\delta}\frac{1}{6} = 0.$$

If $t \equiv 1, 5, 7, 11 \pmod{12}$, then we can apply Theorem 1 (since $\frac{t^2-1}{12}$ is even for those values of t; if $t \equiv 3 \pmod{12}$, then we can apply Theorem 1 to $F_t(3\tau)$. For the even values of t, if $t \equiv 0, 6 \pmod{12}$, then we can apply Theorem 1 to $F_t(24\tau)$; and if $t \equiv 2, 4, 8, 10 \pmod{12}$, then we can apply Theorem 1 to $F_t(8\tau)$.

In particular, for $t = 5$,

$$\begin{aligned} F_5(\tau) &= q \sum_{n=0}^{\infty} \mathrm{asc}_5(n)q^n \\ &= \eta_{4,2}^{1/2}\, \eta_{2,1}^{-1/2}\, \eta_{10,5}^{1/2}\, \eta_{20,10}^{-1/2}\, \eta_{10,0}(\tau) \end{aligned}$$

is a modular form of weight 1 on $\Gamma_0(20)$ with multiplier $i^{a-1}\left(\frac{a}{20}\right)$. Note that $\theta^2(\tau)$ and $\theta^2(5\tau)$ are also modular forms on $\Gamma_0(20)$ with multiplier $i^{a-1}\left(\frac{a}{20}\right)$. Since the $q-$expansions of $F_5(\tau)$ and $\frac{1}{4}(\theta^2(\tau) - \theta^2(5\tau)$ agree for the first terms:

we have

$$q \sum_{n=0}^{\infty} \mathrm{asc}_5(n)q^n = \frac{1}{4}\Big(\theta^2(\tau) - \theta^2(5\tau)\Big),$$

which gives equation (7.3) of [6]:

$$\mathrm{asc}_5(n-1) = \frac{1}{4}\Big[s_2(n) - s_2(n/5)\Big]$$

where it is understood that $s_2(n/5)$ is zero if $n/5$ is not an integer.

Finally, we consider generalized Frobenius partitions. A generalized Frobenius partition of n is a two-rowed array of non-negative integers

$$\begin{pmatrix} a_1 & a_2 & \dots & a_r \\ b_1 & b_2 & \dots & b_r \end{pmatrix}$$

where $a_1 \geq a_2 \geq \cdots \geq a_r \geq 0$, $b_1 \geq b_2 \geq \cdots \geq b_r \geq 0$, and $n = r + \sum_{i=1}^{r} a_i + \sum_{i=1}^{r} b_i$. In [2], Andrews considers two classes of Frobenius partitions. The first class consists of those Frobenius partitions which allow at most k repititions of an integer in either row. Let $\phi_k(n)$ denote the number of such Frobenius partitions of n; then the partition

$$\begin{pmatrix} 5 & 3 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

is counted by $\phi_3(16)$, but not by $\phi_2(16)$. Let $\Phi_k(q) = 1 + \sum_{N=1}^{\infty} \phi_k(n) q^n$. The second class consists of Frobenius partitions of n in which the integers $a_1, a_2, \dots, a_r$ (and, separately, $b_1, b_2, \dots, b_r$) are distinct and chosen from k different copies of the non-negative integers. Each copy can be thought of as having different colors, and hence, these are sometimes called Frobenius partitions with k colors. If there are three colors, then we can use subscripts to denote the copies and define $i_j < k_l$ if $i < k$ or if $i = k$ and $j < l$. Let $c\phi_k(n)$ denote the number of such Frobenius partitions of n; then the partition

$$\begin{pmatrix} 2_3 & 2_2 & 2_1 \\ 2_2 & 1_3 & 1_1 \end{pmatrix}$$

is counted by $c\phi_3(13)$, but not by $c\phi_2(13)$. Let $C\Phi_k(q) = 1 + \sum_{N=1}^{\infty} c\phi_k(n) q^n$.

In [2], Andrews shows how to write the generating functions $\Phi_k(q)$ and $C\Phi_k(q)$

as $q-$products. For example,

$$\Phi_1(q) = C\Phi_1(q) = \prod_{m=1}^{\infty}(1-q^m)^{-1}$$

$$\Phi_2(q) = \prod_{m=1}^{\infty}(1-q^m)^{-1}(1-q^{12m-10})^{-1}(1-q^{12m-9})^{-1}(1-q^{12m-3})^{-1}(1-q^{12m-2})^{-1}$$

$$\Phi_3(q) = \prod_{m=1}^{\infty}(1-q^{12m-6})(1-q^{6m-5})^{-1}(1-q^{6m-4})^{-2}(1-q^{6m-3})^{-3}(1-q^{6m-2})^{-2}(1-q^{6m-1})^{-1}(1-q^{12m})^{-1}$$

$$C\Phi_2(q) = \prod_{m=1}^{\infty}(1-q^{4m-2})(1-q^{2m-1})^{-4}(1-q^{4m})^{-1}$$

$$\begin{aligned} C\Phi_3(q) = &\prod_{m=1}^{\infty}(1-q^{12m})(1-q^{12m-6})^{3}(1-q^{6m-5})^{-5}(1-q^{6m-1})^{-5} \\ &\cdot(1-q^{4m})^{-2}(1-q^{6m-3})^{-7} \\ &+4q\prod_{m=1}^{\infty}(1-q^{12m})(1-q^{4m})(1-q^{12m-6})^{-1} \\ &\cdot(1-q^{4m-2})^{-1}(1-q^m)^{-3}. \end{aligned}$$

We can write these as

$$\Phi_1(q) = C\Phi_1(q) = q^{1/24}\eta_{1,0}^{-1/2}(\tau)$$

$$\Phi_2(q) = q^{1/12}\eta_{1,0}^{-1/2}\,\eta_{12,2}^{-1}\,\eta_{12,3}^{-1}(\tau)$$

$$\Phi_3(q) = q^{1/8}\eta_{12,6}^{-1/2}\,\eta_{6,1}^{-1}\,\eta_{6,2}^{-2}\,\eta_{6,3}^{-3/2}\,\eta_{12,0}^{-1/2}(\tau)$$

$$C\Phi_2(q) = q^{1/12}\eta_{4,2}^{1/2}\,\eta_{2,1}^{-2}\,\eta_{4,0}^{-1/2}(\tau)$$

$$\begin{aligned} C\Phi_3(q) = &\,q^{1/8}\eta_{12,0}^{1/2}\,\eta_{12,6}^{3/2}\,\eta_{6,1}^{-5}\,\eta_{4,0}^{-1}\,\eta_{6,3}^{-7/2}(\tau) \\ &+4q^{1/8}\eta_{12,0}^{1/2}\,\eta_{4,0}^{1/2}\,\eta_{12,6}^{-1/2}\,\eta_{4,2}^{-1/2}\,\eta_{3,0}^{-3/2}(\tau). \end{aligned}$$

In [2], Andrews proves that, for k prime, $c\phi_k(n) \equiv p(n/k) \pmod{k^2}$, (where $p(n/k)$ is taken to be zero if $n/k \notin \mathsf{Z}$). Numerical evidence seems to suggest more; for example,

$$n \equiv 0 (\text{mod } k) \Rightarrow c\phi_k(n) \equiv p(n/k) (\text{mod } k^3).$$

We can show this explicitly for $k = 3$: note that

$$\sum_{n=1}^{\infty}(c\phi_k(n) - p(n/k))q^n = q^{1/8}\eta_{12,0}^{1/2}\,\eta_{12,6}^{3/2}\,\eta_{6,1}^{-5}\,\eta_{4,0}^{-1}\,\eta_{6,3}^{-7/2}(\tau)$$
$$+ 4q^{1/8}\eta_{12,0}^{1/2}\,\eta_{4,0}^{1/2}\,\eta_{12,6}^{-1/2}\,\eta_{4,2}^{-1/2}\,\eta_{3,0}^{-3/2}(\tau) - q^{1/8}\eta_{3,0}^{-1/2}(\tau).$$

The functions

$$\eta_{12,0}^{1/2}\,\eta_{12,6}^{3/2}\,\eta_{6,1}^{-5}\,\eta_{4,0}^{-1}\,\eta_{6,3}^{-7/2}\,\eta_{3,0}^{9/2}(\tau),$$
$$\eta_{12,0}^{1/2}\,\eta_{4,0}^{1/2}\,\eta_{12,6}^{-1/2}\,\eta_{4,2}^{-1/2}\,\eta_{3,0}^{3}(\tau),$$
$$\eta_{3,0}^{4}(\tau), \text{ and } \eta_{9,0}^{3/2}\,\eta_{3,0}^{4}\,\eta_{1,0}^{-3/2}(\tau)$$

are all modular forms of weight 4 on $\Gamma_0(36)$ (by Theorem 1 and Theorem R). Looking at the first few coefficients, we find that

$$\eta_{12,0}^{1/2}\,\eta_{12,6}^{3/2}\,\eta_{6,1}^{-5}\,\eta_{4,0}^{-1}\,\eta_{6,3}^{-7/2}\,\eta_{3,0}^{9/2}(\tau) + 4\eta_{12,0}^{1/2}\,\eta_{4,0}^{1/2}\,\eta_{12,6}^{-1/2}\,\eta_{4,2}^{-1/2}\,\eta_{3,0}^{3}(\tau) - \eta_{3,0}^{4}(\tau)$$
$$= 9q^2 + 27q^3 + 81q^4 + 162q^5 + 351q^6 + 648q^7 + \dots$$
$$= 9\,\eta_{9,0}^{3/2}\,\eta_{3,0}^{4}\,\eta_{1,0}^{-3/2}(\tau).$$

Dividing by $\eta_{3,0}^{9/2}(\tau)$, we get

$$\eta_{12,0}^{1/2}\,\eta_{12,6}^{3/2}\,\eta_{6,1}^{-5}\,\eta_{4,0}^{-1}\,\eta_{6,3}^{-7/2}(\tau) + 4\eta_{12,0}^{1/2}\,\eta_{4,0}^{1/2}\,\eta_{12,6}^{-1/2}\,\eta_{4,2}^{-1/2}\,\eta_{3,0}^{-3/2}(\tau) - \eta_{3,0}^{-1/2}(\tau)$$
$$= 9\,\eta_{9,0}^{3/2}\,\eta_{3,0}^{-1/2}\,\eta_{1,0}^{-3/2}(\tau),$$

or, after multiplying by $q^{1/8}$,

$$\Phi_3(q) - \Phi(3q) = \sum_{n=1}^{\infty}(c\phi_3(n) - p(n/3))q^n = 9q\prod_{m=1}^{\infty}\frac{(1-q^{9m})^3}{(1-q^{3m})(1-q^m)^3}.$$

(Note that if we write the product $\prod \frac{(1-q^{9m})^3}{(1-q^{3m})(1-q^m)^3}$ as a series, then the coefficients will be integers, and hence $c\phi_k(n) - p(n/k)$ is divisible by 9.) Reducing the right-hand side of the above equation modulo 27 is equivalent to reducing $q\prod \frac{(1-q^{9m})^3}{(1-q^{3m})(1-q^m)^3}$ modulo 3, which is

$$q\prod_{m=1}^{\infty}\frac{(1-q^{9m})^3}{(1-q^{3m})(1-q^m)^3}$$
$$= q\prod_{m=1}^{\infty}(1 - 3q^{9m} + 3q^{18m} - q^{27m})(1 + q^{3m} + q^{6m} + q^{9m} + \dots)$$
$$\cdot (1 + 3q^m + 6q^{2m} + 10q^{3m} + \dots + \binom{n+2}{2}q^{nm} + \dots).$$

Since $3 \mid \binom{n+2}{2}$ if and only if $3 \nmid n$, we have

$$q \prod_{m=1}^{\infty} \frac{(1-q^{9m})^3}{(1-q^{3m})(1-q^m)^3} \equiv q \prod_{m=1}^{\infty} (1-q^{27m})(1+q^{3m}+q^{6m}+q^{9m}+\dots) \cdot (1+10q^{3m}+28q^{6m}+\dots)$$
$$\equiv q \prod_{m=1}^{\infty} (1-q^{27m})(1+q^{3m}+q^{6m}+q^{9m}+\dots) \cdot (1+q^{3m}+q^{6m}+q^{9m}+\dots) \pmod 3.$$

The only coefficients which appear in this last product will be as coefficients of q^{3M+1}; hence,

$$n \not\equiv 1 \pmod 3 \Rightarrow c\phi_3(n) \equiv p(n/3) \pmod{27}.$$