

IV. A Generalization of Theorem 1

We now return to Theorem 1 and consider what happens if the exponents $r_{\delta,g}$ are allowed to be half-integers. We will look at the cases of $r_{\delta,0}$ and $r_{\delta,g}$ ($g \neq 0$) separately.

Theorem 4. Let

$$f(\tau) = \prod_{\substack{\delta|N \\ 0 < g < \delta}} \eta_{\delta,g}^{r_{\delta,g}}(\tau)$$

where the $r_{\delta,0}$ are allowed to be half-integers, $r_{\delta,g} \in \mathbb{Z}$ for $g \neq 0$, $r_{\delta,ag} = r_{\delta,g}$ for all a with $(a, N) = 1$; let $k = \sum_{\delta|N} r_{\delta,0} \in \mathbb{Z}$ and $s = \prod_{\delta|N} \delta^{2r_{\delta,0}}$.

If

$$\sum_{\substack{\delta|N \\ 0 < g < \delta}} r_{\delta,g} \delta P_2\left(\frac{g}{\delta}\right) \equiv 0 \pmod{2}$$

and

$$\sum_{\substack{\delta|N \\ 0 < g < \delta}} r_{\delta,g} \frac{N}{\delta} \frac{1}{6} \equiv 0 \pmod{2}$$

then f is modular of weight k on $\Gamma_0(N)$ with multiplier determined by the following transformation formula:

$$\text{For } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \text{ with } (a, 6) = 1,$$

$$f(A\tau) = i^{(a-1)k} \left(\frac{s}{a}\right) (c\tau + d)^k f(\tau).$$

(Cf. Theorem 1 of [7].)

Proof. As in the proof of Theorem 1, we will assume that $r_{\delta, \frac{\delta}{2}} = 0$ for all even δ .

We proceed as before until we get to the point where we need to show

$$\sum_{\substack{\delta|N \\ 0 < g < \delta}} 2r_{\delta,g} a s \left(\frac{c}{\delta}, a; \frac{g}{\delta}, 0\right) \equiv 0 \pmod{2}.$$

For $g \neq 0$, we still have

$$\begin{aligned} \sum_{\substack{\delta|N \\ 0 < g < \delta}} 2r_{\delta,g} a s \left(\frac{c}{\delta}, a; \frac{g}{\delta}, 0\right) &\equiv \sum_{\substack{\delta|N \\ 0 < g < \delta}} \left(\left\lfloor \frac{ag}{\delta} \right\rfloor - \frac{a-1}{2} \right) \\ &\equiv 0 \pmod{2}. \end{aligned}$$

For $g = 0$, we will use the fact that

$$3a s\left(\frac{c}{\delta}, a\right) - \frac{1}{4}(a-1)(a-2) \equiv \frac{1}{2}\left[1 - \left(\frac{c/\delta}{a}\right)\right] \pmod{2}$$

(Theorem 4 of [20]). From this congruence, we find

$$\begin{aligned} 3 \sum_{\delta|N} 2a r_{\delta,0} s\left(\frac{c}{\delta}, a\right) &\equiv \sum_{\delta|N} r_{\delta,0} \left[1 - \left(\frac{c/\delta}{a}\right)\right] + \frac{(a-1)(a-2)}{2} \sum_{\delta|N} r_{\delta,0} \\ &= \sum_{\delta|N} r_{\delta,0} \left[1 - \left(\frac{c/\delta}{a}\right)\right] + \frac{(a-1)(a-2)}{2} k \pmod{2}. \end{aligned}$$

Since $r_{\delta,0} \left[1 - \left(\frac{c/\delta}{a}\right)\right]$ and $\frac{(a-1)}{2}$ are integers, both sides of the above congruence are integers. This implies that

$$\begin{aligned} \sum_{\delta|N} 2a r_{\delta,0} s\left(\frac{c}{\delta}, a\right) &\equiv 3 \sum_{\delta|N} 2a r_{\delta,0} s\left(\frac{c}{\delta}, a\right) \\ &\equiv \sum_{\delta|N} r_{\delta,0} \left[1 - \left(\frac{c/\delta}{a}\right)\right] + \frac{(a-1)(a-2)}{2} \sum_{\delta|N} r_{\delta,0} \pmod{2}. \end{aligned}$$

The remainder of the proof, showing that

$$\sum_{\substack{\delta|N \\ 0 < g < \delta}} r_{\delta,g} \left[b\delta P_2\left(\frac{g}{\delta}\right) - \frac{c}{6\delta} + 2a s\left(\frac{c}{\delta}, a; \frac{g}{\delta}, 0\right) \right] \equiv 0 \pmod{a}$$

proceeds as in the proof of Theorem 1.

Thus, the multiplier in the transformation formula is

$$\begin{aligned} \exp\left(\pi i \sum_{\delta|N} 2a r_{\delta,0} s\left(\frac{c}{\delta}, a\right)\right) &= \exp\left(\pi i \left[\sum_{\delta|N} r_{\delta,0} \left[1 - \left(\frac{c/\delta}{a}\right)\right] + \frac{a-1}{2}(a-2)k \right]\right) \\ &= \prod_{\delta|N} \exp\left(\pi i \left[r_{\delta,0} \left[1 - \left(\frac{c/\delta}{a}\right)\right] \right]\right) \exp\left(\pi i \left[\frac{a-1}{2}k \right]\right) \\ &= \prod_{\delta|N} \left(\frac{c/\delta}{a}\right)^{2r_{\delta,0}} i^{(a-1)k} \\ &= \prod_{\delta|N} \left(\frac{c}{a}\right)^{2r_{\delta,0}} \left(\frac{\delta}{a}\right)^{2r_{\delta,0}} i^{(a-1)k} \\ &= \left(\frac{c}{a}\right)^{2k} \left(\frac{s}{a}\right) i^{(a-1)k} \\ &= \left(\frac{s}{a}\right) i^{(a-1)k}, \end{aligned}$$

as claimed. \square

As an example, consider the function $f(\tau) = \eta_{5,0}^{5/2} \eta_{1,0}^{-1/2}(\tau) = q \prod_{n=1}^{\infty} \frac{(1-q^{5n})^5}{(1-q^n)}$.

For $N = 5$, we have

$$\sum_{\substack{\delta|N \\ 0 < g < \delta}} r_{\delta,g} \delta P_2\left(\frac{g}{\delta}\right) = \frac{5}{2} \cdot 5 \cdot \frac{1}{6} - \frac{1}{2} \cdot \frac{1}{6} = 2$$

and

$$\sum_{\substack{\delta|N \\ 0 < g < \delta}} r_{\delta,g} \frac{N}{6\delta} = \frac{5}{2} \cdot \frac{5}{6 \cdot 5} - \frac{1}{2} \cdot \frac{5}{6} = 0.$$

Since $s = \prod_{\delta|5} \delta^{2r_{\delta,0}} = 5^5 1^{-1} = 5^5$ and $k = \sum_{\delta|N} r_{\delta,0} = 2$, we find that

$$\begin{aligned} f(a\tau) &= i^{2(a-1)} \left(\frac{5^5}{a}\right) (c\tau + d)^2 f(\tau) \\ &= \left(\frac{5}{a}\right) (c\tau + d)^2 f(\tau). \end{aligned}$$

Checking the order of f at the cusps (order 1 at infinity, order 0 at zero), we conclude that f is a modular form of weight 2 on $\Gamma_0(5)$ with the Legendre symbol $\left(\frac{\cdot}{5}\right)$ as its multiplier, i.e. $f \in M_2(\Gamma_0(5), \left(\frac{\cdot}{5}\right))$.

Using the transformation formulas for $\eta_{5,1}(\tau)$ and $\eta_{5,2}(\tau)$,

$$\eta_{5,1}(A\tau) = \nu_{5,1}(A) \eta_{5,a}(\tau)$$

$$\eta_{5,2}(A\tau) = \nu_{5,2}(A) \eta_{5,2a}(\tau)$$

we find that

$$\frac{\eta'_{5,1}}{\eta_{5,1}}(A\tau) = (c\tau + d)^2 \frac{\eta'_{5,a}}{\eta_{5,a}}(\tau)$$

and

$$\frac{\eta'_{5,2}}{\eta_{5,2}}(A\tau) = (c\tau + d)^2 \frac{\eta'_{5,2a}}{\eta_{5,2a}}(\tau).$$

If $a \equiv \pm 1 \pmod{5}$, then

$$\frac{\eta'_{5,1}}{\eta_{5,1}}(A\tau) - \frac{\eta'_{5,2}}{\eta_{5,2}}(A\tau) = (c\tau + d)^2 \left[\frac{\eta'_{5,1}}{\eta_{5,1}}(\tau) - \frac{\eta'_{5,2}}{\eta_{5,2}}(\tau) \right];$$

if $a \equiv \pm 2 \pmod{5}$, then

$$\frac{\eta'_{5,1}}{\eta_{5,1}}(A\tau) - \frac{\eta'_{5,2}}{\eta_{5,2}}(A\tau) = (c\tau + d)^2 \left[\frac{\eta'_{5,2}}{\eta_{5,2}}(\tau) - \frac{\eta'_{5,1}}{\eta_{5,1}}(\tau) \right].$$

Since $\frac{\eta'_{5,1}}{\eta_{5,1}}(\tau)$ and $\frac{\eta'_{5,2}}{\eta_{5,2}}(\tau)$ are modular forms of weight 2 on $\Gamma_1(5)$ (and hence are holomorphic at the cusps), $\frac{\eta'_{5,1}}{\eta_{5,1}}(\tau) - \frac{\eta'_{5,2}}{\eta_{5,2}}(\tau)$ is a modular form of weight 2 on $\Gamma_0(5)$ with the Legendre symbol $\left(\frac{\cdot}{5}\right)$ as its multiplier.

Looking at the q -expansions of $f(\tau) = \eta_{5,0}^{5/2} \eta_{1,0}^{-1/2}(\tau)$ and $\frac{\eta'_{5,1}}{\eta_{5,1}}(\tau) - \frac{\eta'_{5,2}}{\eta_{5,2}}(\tau)$,

$$f(\tau) = q + q^2 + 2q^3 + \dots$$

$$\frac{\eta'_{5,1}}{\eta_{5,1}}(\tau) - \frac{\eta'_{5,2}}{\eta_{5,2}}(\tau) = 2\pi i \left(\frac{1}{5} - q + q^2 + 2q^3 + \dots \right),$$

we see that these modular forms are linearly independent. In fact, they form a basis of $M_2(\Gamma_0(5), \left(\frac{\cdot}{5}\right))$. This can be deduced from the following:

Proposition. Let χ be a character of order m on Γ' where Γ' is a congruence subgroup of $\Gamma = SL_2(\mathbb{Z})$ of level N , and let $n \in \mathbb{N}$. Then

$$\dim(M_n(\Gamma', \chi)) \square \dim(M_{mn}(\Gamma')).$$

Proof. Suppose f_1, f_2, \dots, f_d are linearly independent modular forms on $M_n(\Gamma', \chi)$.

Then the functions

$$f_1^m, f_2 f_1^{m-1}, \dots, f_d f_1^{m-1}$$

are linearly independent modular forms on $M_{mn}(\Gamma')$, giving the desired inequality. \square

Since $\left(\frac{\cdot}{5}\right)$ is a quadratic character modulo 5, we have

$$\dim(M_2(\Gamma_0(5), \left(\frac{\cdot}{5}\right))) \square \dim(M_4(\Gamma_0(5))) = 2.$$

Thus, $\eta_{5,0}^{5/2} \eta_{1,0}^{-1/2}(\tau)$ and $\frac{\eta'_{5,1}}{\eta_{5,1}}(\tau) - \frac{\eta'_{5,2}}{\eta_{5,2}}(\tau)$ form a basis for $M_2(\Gamma_0(5), \left(\frac{\cdot}{5}\right))$.

For the next result, we adopt the following notation: let r denote a number which is a prime to some power; for example, the sum $\sum_{a=r}^{\alpha}$ is interpreted as being taken over all a such that $a = p^\alpha$, where p is prime and $\alpha > 1$.

Theorem 5. Let

$$f(\tau) = \prod_{\substack{\delta|N \\ 0 < g < \delta}} \eta_{\delta,g}^{r_{\delta,g}}(\tau)$$

where the $r_{\delta,g}$ are allowed to be half-integers and $r_{\delta,ag} = r_{\delta,g}$ for all a with $(a, N) = 1$;

let $s = \prod_{\delta|N} \delta^{2r_{\delta,0}}$, $t = \prod_{\delta|N} \prod_{\substack{d|\delta \\ \delta/d=r}} p^{2r_{\delta,d}}$ and $k = \sum_{\delta|N} r_{\delta,0} \in \mathbb{Z}$.

If

$$\sum_{\substack{\delta|N \\ 0 < g < \delta}} r_{\delta,g} \delta P_2\left(\frac{g}{\delta}\right) \equiv 0 \pmod{2}$$

and

$$\sum_{\substack{\delta|N \\ 0 < g < \delta}} r_{\delta,g} \frac{N}{\delta} \frac{1}{6} \equiv 0 \pmod{2}$$

then f is modular of weight k on $\Gamma_0(N)$ with multiplier determined by the following transformation formula:

$$\text{For } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \text{ with } (a, 6) = 1,$$

$$f(A\tau) = i^{(a-1)k} \left(\frac{s}{a}\right) \left(\frac{t}{a}\right) (c\tau + d)^k f(\tau).$$

Proof. Again, we assume that $r_{\delta, \frac{\delta}{2}} = 0$. Proceeding as in the proof of Theorem 1, we need to consider the value of $\sum_{\substack{\delta|N \\ 0 < g < \delta}} 2r_{\delta,g} a s\left(\frac{c}{\delta}, a; \frac{g}{\delta}, 0\right)$ modulo 2. Using the fact that $r_{\delta,ag} = r_{\delta,g}$ for all a with $(a, N) = 1$, we get

$$\begin{aligned} \sum_{\substack{\delta|N \\ 0 < g < \delta}} 2r_{\delta,g} a s\left(\frac{c}{\delta}, a; \frac{g}{\delta}, 0\right) &= \sum_{\delta|N} \sum_{d|\delta} \sum_{(g,\delta)=d} 2r_{\delta,da} s\left(\frac{c}{\delta}, a; \frac{g}{\delta}, 0\right) \\ &= \sum_{\delta|N} \sum_{d|\delta} 2r_{\delta,da} \sum_{e|\frac{\delta}{d}} \mu(e) s\left(\frac{c}{de}, a\right) \\ &= \sum_{\delta|N} \sum_{d|\delta} 2r_{\delta,da} \sum_{\frac{\delta}{d}=ef} \mu(e) s\left(\frac{cf}{\delta}, a\right). \end{aligned}$$

By Theorem 4 of [20],

$$3a s\left(\frac{cf}{\delta}, a\right) \equiv \frac{1}{4}(a-1)(a-2) + \frac{1}{2}\left(1 - \left(\frac{cf/\delta}{a}\right)\right);$$

multiply by $2r_{\delta,d}$ (which is an integer) and we have

$$3 \cdot 2ar_{\delta,d}s\left(\frac{cf}{\delta}, a\right) \equiv \frac{r_{\delta,d}}{2}(a-1)(a-2) + r_{\delta,d}\left(1 - \left(\frac{cf/\delta}{a}\right)\right).$$

So,

$$\begin{aligned} 3 \sum_{(g,\delta)=d} 2r_{\delta,d}as\left(\frac{c}{\delta}, a; \frac{g}{\delta}, 0\right) &= 3 \cdot 2r_{\delta,d}a \sum_{\frac{\delta}{a}=ef} \mu(e)s\left(\frac{cf}{\delta}, a\right). \\ &\equiv \sum_{\frac{\delta}{a}=ef} \mu(e) \left[\frac{r_{\delta,d}}{2}(a-1)(a-2) + r_{\delta,d}\left(1 - \left(\frac{cf/\delta}{a}\right)\right) \right] \\ &= r_{\delta,d} \frac{(a-1)(a-2)}{2} \sum_{\frac{\delta}{a}=ef} \mu(e) + r_{\delta,d} \sum_{\frac{\delta}{a}=ef} \mu(e) \left[1 - \left(\frac{cf/\delta}{a}\right) \right] \\ &= r_{\delta,d} \sum_{\frac{\delta}{a}=ef} \mu(e) \left[1 - \left(\frac{cf/\delta}{a}\right) \right] \pmod{2}. \end{aligned}$$

This last quantity is an integer, so we can write

$$\sum_{(g,\delta)=d} 2r_{\delta,d}as\left(\frac{c}{\delta}, a; \frac{g}{\delta}, 0\right) \equiv r_{\delta,d} \sum_{\frac{\delta}{a}=ef} \mu(e) \left[1 - \left(\frac{cf/\delta}{a}\right) \right] \pmod{2}.$$

To find the value of $r_{\delta,d} \sum_{\frac{\delta}{a}=ef} \mu(e) \left[1 - \left(\frac{cf/\delta}{a}\right) \right]$ modulo 2, we use Lemma 2:

$$\begin{aligned} r_{\delta,d} \sum_{\frac{\delta}{a}=ef} \mu(e) \left[1 - \left(\frac{cf/\delta}{a}\right) \right] &= r_{\delta,d} \sum_{\frac{\delta}{a}=ef} \mu(e) - r_{\delta,d} \sum_{\frac{\delta}{a}=ef} \mu(e) \left(\frac{cf/\delta}{a}\right) \\ &\equiv r_{\delta,d} \sum_{\frac{\delta}{a}=ef} \mu(e) \left(\frac{cf/\delta}{a}\right) \\ &= r_{\delta,d} \left(\frac{c/\delta}{a}\right) \sum_{\frac{\delta}{a}=ef} \mu(e) \left(\frac{f}{a}\right); \end{aligned}$$

by Lemma 2, this last term can only be odd when $\frac{\delta}{a} = p^\alpha$ for some prime p with $\left(\frac{p}{a}\right) = -1$, in which case we get

$$\begin{aligned} r_{\delta,d} \sum_{\frac{\delta}{a}=ef} \mu(e) \left[1 - \left(\frac{cf/\delta}{a}\right) \right] &= r_{\delta,d} \left(\frac{c/\delta}{a}\right) \sum_{\frac{\delta}{a}=ef} \mu(e) \left(\frac{f}{a}\right) \\ &= r_{\delta,d} \left(\frac{c/\delta}{a}\right) \left[\left(\frac{p}{a}\right)^\alpha - \left(\frac{p}{a}\right)^{\alpha-1} \right] \\ &= r_{\delta,d} \left(\frac{c/\delta}{a}\right) \left(\frac{\delta/d}{a}\right) \left[1 - \left(\frac{p}{a}\right) \right] \\ &\equiv 2r_{\delta,d} \pmod{2}. \end{aligned}$$

Therefore,

$$\sum_{\delta|N} \sum_{d|\delta} \sum_{(g,\delta)=d}^* 2r_{\delta,d} a s\left(\frac{c}{\delta}, a; \frac{g}{\delta}, 0\right) \equiv \sum_{\delta|N} \sum_{d|\delta}^* 2r_{\delta,d} \pmod{2}$$

where the sum \sum^* is taken only over those divisors d of δ for which $\frac{\delta}{d} = p^\alpha$ with $\left(\frac{p}{a}\right) = -1$. Since $\sum_{\delta|N} \sum_{d|\delta} \sum_{(g,\delta)=d}^* 2r_{\delta,d} a s\left(\frac{c}{\delta}, a; \frac{g}{\delta}, 0\right)$ is an integer, the remainder of the proof (showing that $\sum_{\substack{\delta|N \\ 0 \leq g < \delta}} r_{\delta,g} \left[b\delta P_2\left(\frac{g}{\delta}\right) - \frac{c}{6\delta} + 2as\left(\frac{c}{\delta}, a; \frac{g}{\delta}, 0\right) \right] \equiv 0 \pmod{a}$) proceeds as before.

The multiplier for f then is

$$\begin{aligned} & \exp\left(\pi i \sum_{\substack{\delta|N \\ 0 \leq g < \delta}} 2a s\left(\frac{c}{\delta}, a; \frac{g}{\delta}, 0\right)\right) \\ &= \exp\left(\pi i \sum_{\substack{\delta|N \\ g=0}} 2a s\left(\frac{c}{\delta}, a\right)\right) \exp\left(\pi i \sum_{\substack{\delta|N \\ 0 \leq g < \delta}} 2a s\left(\frac{c}{\delta}, a; \frac{g}{\delta}, 0\right)\right) \\ &= \left(\frac{s}{a}\right) i^{(a-1)k} \exp\left(\pi i \sum_{\delta|N} \sum_{d|\delta}^* 2r_{\delta,d}\right) \\ &= \left(\frac{s}{a}\right) i^{(a-1)k} \prod_{\delta|N} \prod_{d|\delta}^* \left(\frac{p}{a}\right)^{2r_{\delta,d}} \\ &= \left(\frac{s}{a}\right) i^{(a-1)k} \prod_{\delta|N} \prod_{\substack{d|\delta \\ \delta/d=\tau}} \left(\frac{p}{a}\right)^{2r_{\delta,d}} \\ &= \left(\frac{s}{a}\right) \left(\frac{t}{a}\right) i^{(a-1)k}. \end{aligned}$$

□

Note: We can also write the transformation formula as

$$f(A\tau) = i^{(a-1)k} \left(\frac{s}{a}\right) (c\tau + d)^k f(\tau)$$

where $s = \prod_{\delta|N} \prod_{\substack{d|\delta \\ \delta/d=\tau}} p^{2r_{\delta,0} + 2r_{\delta,d}}$.

From this Theorem, we find that $\theta^2(\tau) = \eta_{2,0} \eta_{4,2}^2 \eta_{2,1}^{-2}(\tau)$ is modular on $\Gamma_0(4)$ of weight 1 with multiplier $i^{a-1} = \left(\frac{-1}{a}\right)$. Recall that

$$\theta(\tau) = \sum_{N=0}^{\infty} s_2(N) q^N = 1 + 4 \sum_{N=1}^{\infty} \left(\sum_{d|N} i^{d-1} \right) q^N$$

([8]). In fact, $\theta^2(\tau)$ is the only modular form on $\Gamma_0(4)$ of weight 1 with this multiplier ([13], Proposition 4 of chapter 4, section 1).

It would be interesting to try to derive a formula for $p_2(N)$ and $o_2(N)$ by looking at

$$\eta_{18,0} \eta_{36,12} \eta_{18,6}^{-2}(\tau) = q^{1/2} \sum_{N=1}^{\infty} p_2(N) q^{6N}$$

and

$$\eta_{18,0} \eta_{36,6} \eta_{18,3}^{-2}(\tau) = \sum_{N=1}^{\infty} o_2(N) q^{3N+2}.$$

The latter function satisfies the congruences in Theorem 5 and is holomorphic at the cusps of $\Gamma_0(36)$ and therefore is a modular form on $\Gamma_0(36)$ of weight 1 with multiplier i^{a-1} . The former function does not satisfy the congruences in Theorem 5 ($\sum r_{\delta,g} \delta P_2(g/\delta) = 1$); but, replacing τ by 2τ , the function $\eta_{36,0} \eta_{72,24} \eta_{36,12}^{-2}(\tau) = \sum_{N=1}^{\infty} p_2(N) q^{12N+1}$, is holomorphic at the cusps of $\Gamma_0(72)$, and hence is a modular form of weight 1 on $\Gamma_0(72)$ with multiplier i^{a-1} .