III. Some Combinatorial Applications

As an easy example, consider $M_2(\Gamma_0(4))$. From [23] (p. 23-25), we find that $\Gamma_0(4)$ has three cusps, no elliptic points, and hence $\mathfrak{h}/\Gamma_0(4)$ has genus $1+\frac{6}{12}-\frac{3}{2}=0$. Therefore, by [9] (Theorem 1 of section 8), this space has dimension 2. Looking at the expansions of $H_2^{(4,2)}(\tau)$ and $H_2^{(4,1)}(\tau)$:

$$\begin{aligned} H_2^{(4,1)}(\tau) &= 1+24\sum_{M=1}^{\infty}\sigma^{(4,1)}(M)q^M \\ &= 1+24q+24q^2+96q^3+\dots, \\ H_2^{(4,2)}(\tau) &= 1+6\sum_{M=1}^{\infty}\sigma^{(4,2)}(M)q^M \\ &= 1+24q^2+\dots, \end{aligned}$$

we see that these two modular forms are linearly independent, and hence form a basis for $M_2(\Gamma_0(4))$.

Recall that $\theta^4(\tau)=\Big(\sum_{n\in Z}q^{n^2}\Big)^4\in M_2(\Gamma_0(4))$ (exercises 10–11 of chapter 3, section 3 in [13]). In fact, $\theta^4(\tau)=\eta_{1,0}^{-4}\,\eta_{2,0}^{10}\,\eta_{4,0}^{-4}(\tau)$ (also from [13]), which satisfies the condition of Theorem 1. If $s_4(M)$ denotes the number of ways of writing M as a sum of four squares, then $\theta^4(\tau)=\sum_{M=1}^{\infty}s_4(M)q^M$. Using the expansion $\theta^4(\tau)=1+8q+24q^2+\dots$, we find that

$$\theta^4(\tau)=\frac{1}{3}H_2^{(4,1)}(\tau)+\frac{2}{3}H_2^{(4,2)}(\tau);$$

equating the coefficients of q^M on both sides of the above eqation, we get

$$s_4(M)=8\sigma^{(4,1)}(M)+4\sigma^{(4,2)}(M)$$

(cf. Corollary on p. 282 of [10]).

Similarly, $F(\tau)=q\Psi^4(2\tau)=\eta_{2,0}^{-2}\,\eta_{4,0}^{4}(\tau)=q\Big(\sum_{n=0}^{\infty}q^{n^2+n}\Big)^4\in M_2(\Gamma_0(4))$ ($\Psi(q)$ is the function $\sum_{n=0}^{\infty}q^{(n^2+n)/2}$ from [19] and the function $q\Psi^4(2\tau)$ is the same as the function F in exercises 10–11 of chapter 3, section 3 in [13]). Note that $\eta_{2,0}^{-2}\,\eta_{4,0}^{4}(\tau)$ satisfies the conditions of Theorem 1. If $t_4(M)$ denotes the number of ways of writing M as a sum of four triangular numbers, then $F(\tau) = \sum_{M=0}^{\infty} t_4(M) q^{2M+1}$. From the expansion $F(\tau) = q + 4q^3 + \dots$, we find that

$$F(\tau) = \frac{1}{24} H_2^{(4,1)}(\tau) - \frac{1}{24} H_2^{(4,2)}(\tau);$$

equating the coefficients of q^{2M+1} on both sides, we get

$$\begin{aligned} t_4(M) &= \sigma^{(4,1)}(2M+1) - \frac{1}{4}\sigma^{(4,2)}(2M+1) \\ &= \sigma(2M+1) \end{aligned}$$

(cf. Theorem 3 of [19]).

These are classical results which have been known for some time. We now try to establish similar results for pentagonal and octagonal numbers using modular forms. In particular, we will prove the following:

$$\text{If } M \equiv 1 \pmod 2, \text{ then } p_4(M) + o_4(2M-1) \equiv 0 \pmod 4 \tag{9}$$

$$\text{If } M \equiv 1 \pmod 4, \text{ then } p_4(M) + o_4(2M-1) \equiv 0 \pmod 8 \tag{10}$$

$$\text{If } M \equiv 1 \pmod 2, \text{ then } o_4(2M) \equiv 0 \pmod 6 \tag{11}$$

$$\text{If } M \in \mathsf{N}, \text{ then } \frac{1}{2}\sigma(6M+1) \Box p_4(M) \Box \sigma(6M+1) \tag{12}$$

$$\text{If } M \equiv 2 \pmod 4, \text{ then } o_4(M) = \frac{1}{3}\sigma(3M+4) \tag{13}$$

$$\text{If } M \equiv 0 \pmod 4, \text{ then } o_4(M) = \frac{1}{3}\sigma(3M+4) - \frac{4}{3}\sigma(\frac{3M+4}{4}) \tag{14}$$

To do this, we establish two formulas.

Theorem 2. Let a, b be half-integers, with $a - b \in \mathsf{Z}$ and $a > |b|$. Then

$$q^{b^2/4a} \sum_{n \in \mathsf{Z}} q^{an^2+bn} = \eta_{2a,0}^{1/2}\, \eta_{4a,2a-2b}\, \eta_{2a,a-b}^{-1}(\tau).$$

Proof. Starting with the Jacobi Triple Product ([1], Theorem 2.8)

$$\sum_{n\in\mathsf{Z}} q^{n^2} z^n = \prod_{n=1}^{\infty}(1-q^{2n})(1+zq^{2n-1})(1+z^{-1}q^{2n-1}),$$

replace q by q^a and z by q^b, and we have

$$\begin{aligned}\sum_{n\in\mathsf{Z}} q^{an^2+bn} &= \prod_{n=1}^{\infty}(1-q^{2an})(1+q^{2an-a+b})(1+q^{2an-a-b})\\ &= \prod_{n=1}^{\infty}(1-q^{2an})\frac{(1-q^{4an-2a+2b})(1-q^{4an-2a-2b})}{(1-q^{2an-a+b})(1-q^{2an-a-b})}\\ &= q^{-b^2/4a}\,\eta_{2a,0}^{1/2}\,\eta_{4a,2a-2b}\,\eta_{2a,a-b}^{-1}(\tau).\end{aligned}$$

$\square$

Note: If we let $a=1, b=0$, then the above Theorem gives $\theta(\tau)=\eta_{2,0}^{1/2}\,\eta_{4,2}\,\eta_{2,1}^{-1}(\tau)$, which can also be written as $\eta_{1,0}^{-1}\,\eta_{2,0}^{5/2}\,\eta_{4,0}^{-1}(\tau)$ (as in [13]). If special care is taken, then we can also see what happens if $a=\frac{1}{2}=b$: replacing q and z in the Jacobi Triple Product by $q^{1/2}$, we get

$$\begin{aligned}\sum_{n\in\mathsf{Z}} q^{\frac{1}{2}n^2+\frac{1}{2}n} &= \prod_{n=1}^{\infty}(1-q^n)(1+q^n)(1+q^{n-1})\\ &= 2\prod_{n=1}^{\infty}(1-q^n)(1+q^n)^2,\end{aligned}$$

or, $2\sum_{n=0}^{\infty} q^{\frac{1}{2}n^2+\frac{1}{2}n} = 2\prod_{n=1}^{\infty}(1-q^n)(1+q^n)^2$. So,

$$\begin{aligned}\sum_{n=0}^{\infty} q^{\frac{1}{2}n^2+\frac{1}{2}n} &= \prod_{n=1}^{\infty}(1-q^n)(1+q^n)^2\\ &= \prod_{n=1}^{\infty}(1-q^n)\frac{(1-q^{2n})^2}{(1-q^n)^2}\\ &= q^{-1/8}\,\eta_{1,0}^{-1/2}\,\eta_{2,0}(\tau),\end{aligned}$$

i.e. $q^{1/8}\sum_{n=0}^{\infty} q^{\frac{1}{2}n^2+\frac{1}{2}n} = \eta_{1,0}^{-1/2}\,\eta_{2,0}(\tau)$. Replacing τ by 2τ (q by q^2) in this equation, and then raising both sides to the fourth power, gives

$$q\Big(\sum_{n=0}^{\infty} q^{n^2+n}\Big)^4 = \eta_{2,0}^{-2}\,\eta_{4,0}^{4}(\tau).$$

Theorem 3. Let a, b be half-integers, with $a - b \in \mathbb{Z}$ and $0 < |b| < a$. Then

$$q^{(2a-3b)^2/12a}\Big[\sum_{n\in\mathbb{Z}} q^{3an^2-(2a-3b)n} - \sum_{n\in\mathbb{Z}} q^{3an^2-(2a+3b)n+2b}\Big]$$
$$= \eta_{2a,0}^{1/2}\eta_{2a,a-b}\,\eta_{4a,2b}(\tau).$$

Proof. This time we start with the Quintuple Product ([18], equation (10))

$$\prod_{n=1}^{\infty}(1-q^{2n})(1-zq^{2n-1})(1-z^{-1}q^{2n-1})(1-z^2q^{4n-4})(1-z^{-2}q^{4n-4})$$
$$= \sum_{n\in\mathbb{Z}} q^{3n^2-2n}(z^{3n}+z^{-3n}-z^{3n-2}-z^{-3n+2}),$$

and replace q by q^a and z by q^b:

$$\prod_{n=1}^{\infty}(1-q^{2an})(1-q^{2an-a+b})(1-q^{2an-a-b})(1-q^{4an-4a+2b})(1-q^{4an-4a-2b})$$
$$= \sum_{n\in\mathbb{Z}} q^{3an^2-2an}(q^{3bn}+q^{-3bn}-q^{3bn-2b}-q^{-3bn+2b})$$
$$= \sum_{n\in\mathbb{Z}} q^{3an^2-2an}(q^{3bn}-q^{-3bn+2b})(1-q^{-2b}).$$

Cancelling $1 - q^{-2b}$ from both sides of the above equation and using

$$\frac{1}{2}2aP_2(0) + 2aP_2\Big(\frac{a+b}{2a}\Big) + 4aP_2\Big(\frac{b}{2a}\Big) = \frac{(2a-3b)^2}{6a},$$

we get the desired result. □

Using Theorem 2, we derive two formulas of interest:

$$a = \frac{3}{2} \quad b = \frac{1}{2}: \qquad q^{1/24}\sum_{n\in\mathbb{Z}} q^{\frac{3}{2}n^2+\frac{1}{2}n} = \eta_{3,0}^{1/2}\,\eta_{6,2}\,\eta_{3,1}^{-1}(\tau) \tag{15}$$

$$a = 3 \quad b = 2: \qquad q^{1/3}\sum_{n\in\mathbb{Z}} q^{3n^2+2n} = \eta_{6,0}^{1/2}\,\eta_{12,2}\,\eta_{6,1}^{-1}(\tau), \tag{16}$$

which will be used later to construct modular forms. These formulas are of interest, because they can be viewed as generating functions for figurate numbers; specifically, $\frac{3}{2}n^2 - \frac{1}{2}n$ is the formula for pentagonal numbers and $3n^2 - 2n$ is the formula for octagonal numbers. In the above sums, the functions generate pentagonal/octagonal numbers of general rank, e.g. $\frac{3}{2}(2)^2 - \frac{1}{2}(2) = 5$ is a pentagonal number of positive rank, $\frac{3}{2}(-2)^2-\frac{1}{2}(-2)=7$ is a pentagonal number of negative rank, and both appear as exponents of q in the expansion of $\sum_{n\in\mathbb{Z}} q^{\frac{3}{2}n^2+\frac{1}{2}n}$.

We can also consider the case $a=2,\ b=1$; however, this gives the generating function for hexagonal numbers, which turns out to be the same as triangular numbers.

Using $a=2,\ b=1$ in Theorem 3, we get the following formula:

$$a=2 \quad b=1: \qquad q^{1/24}\Big(\sum_{n\in\mathbb{Z}} q^{9n^2-3n}-\sum_{n\in\mathbb{Z}} q^{9n^2+9n+2}\Big)=\eta_{4,0}^{1/2}\,\eta_{4,1}\,\eta_{8,2}(\tau),$$

or, since $\eta_{4,0}^{1/2}\,\eta_{4,1}\,\eta_{8,2}(\tau)=\eta_{1,0}^{1/2}(\tau)$,

$$\eta_{1,0}^{1/2}(\tau)=q^{1/24}\sum_{n\in\mathbb{Z}}(-1)^n q^{\frac{3}{2}n^2-\frac{1}{2}n}. \tag{17}$$

This last equation is Euler's Pentagonal Number Theorem (Cor. 1.7 of [1]).

Now, $\eta_{3,0}^{1/2}\,\eta_{6,2}\,\eta_{3,1}^{-1}(\tau)$ (from (15)) is not a modular form, but it can be used to construct one. First, since $\eta_{3,0}^{1/2}\,\eta_{6,2}\,\eta_{3,1}^{-1}(\tau)=q^{1/24}\sum_{n\in\mathbb{Z}} q^{\frac{3}{2}n^2+\frac{1}{2}n}$, we have

$$\eta_{3,0}^{2}\,\eta_{6,2}^{4}\,\eta_{3,1}^{-4}(\tau)=q^{1/6}\sum_{M=0}^{\infty} p_4(M)q^M,$$

where $p_4(M)$ denotes the number of ways of writing M as a sum of four pentagonal numbers. Since

$$\sum_{\substack{\delta\mid N\\ 0\le g<\delta}} r_{\delta,g}\delta P_2\Big(\frac{g}{\delta}\Big)=2\cdot 3P_2(0)+4\cdot 6P_2\Big(\frac{2}{6}\Big)-4\cdot 3P_2\Big(\frac{1}{3}\Big)=\frac{1}{3}$$

$$\sum_{\substack{\delta\mid N\\ 0\le g<\delta}} r_{\delta,g}\frac{N}{6\delta}=\frac{N}{6}\Big(\frac{2}{3}+\frac{4}{6}-\frac{4}{3}\Big)=0,$$

if we replace τ by 6τ (q by q^6), then the function

$$P(\tau)=\eta_{3,0}^{2}\,\eta_{6,2}^{4}\,\eta_{3,1}^{-4}(6\tau)=\eta_{18,0}^{2}\,\eta_{36,12}^{4}\,\eta_{18,6}^{-4}(\tau)=\sum_{M=0}^{\infty} p_4(M)q^{6M+1}$$

satisfies the congruences of Theorem 1. Therefore $P(\tau)$ is a modular function of weight 2 on $\Gamma_0(N)$ for any N divisible by 36; in particular, $P(\tau)$ is a modular function of weight 2 on $\Gamma_0(36)$. We can use Theorem R to check that $P(\tau)$ is holomorphic at the cusps of $\Gamma_0(36)$:

for $\epsilon|36$, and $(\lambda, 36) = (\lambda, \mu) = (\mu, 36) = 1$, we get the following orders

$\epsilon = 1:$ order 0

$\epsilon = 2:$ order 0

$\epsilon = 3:$ order 0

$\epsilon = 4:$ order 1

$\epsilon = 6:$ order 0

$\epsilon = 9:$ order 4

$\epsilon = 12:$ order 3

$\epsilon = 18:$ order 8

$\epsilon = 36:$ order 1.

Since the orders are all nonnegative, $P(\tau)$ is holomorphic at all of the cusps of $\Gamma_0(36)$, and hence $P(\tau) \in M_2(\Gamma_0(36))$. (Note: strictly speaking, $P(\tau) = \eta_{18,0}^{2}\,\eta_{36,12}^{4}\,\eta_{18,6}^{-4}(\tau)$ does not satisfy the condition $r_{\delta,ag} = r_{\delta,g}$ for $(a, 36) = 1$; however, this is easily remedied by writing $P(\tau)$ as $\eta_{18,0}^{2}\,\eta_{36,12}^{2}\,\eta_{36,24}^{4}\,\eta_{18,6}^{-2}\,\eta_{18,12}^{-4}(\tau)$.)

Similarly, we can construct a modular form using (16)

$$\eta_{6,0}^{1/2}\,\eta_{12,2}\,\eta_{6,1}^{-1}(\tau) = q^{1/3}\sum_{n\in Z} q^{3n^2+2n}:$$

raise both sides to the fourth power and replace τ by 3τ (q by q^3) and we have

$$O(\tau) = \eta_{18,0}^{2}\,\eta_{36,6}^{4}\,\eta_{18,3}^{-4}(\tau) = \sum_{M=0}^{\infty} o_4(M)q^{3M+4}$$

where $o_4(M)$ denotes the number of ways of writing M as a sum of four octagonal numbers. Notice that $\eta_{18,0}^{2}\,\eta_{36,6}^{4}\,\eta_{18,3}^{-4}(\tau)$ satisfies the congruences of Theorem 1:

$$\sum_{\substack{\delta|N \\ 0\quad g<\delta}} r_{\delta,g}\delta P_2\Big(\frac{g}{\delta}\Big) = 2\cdot 18P_2(0) + 4\cdot 36P_2\Big(\frac{6}{36}\Big) - 4\cdot 18P_2\Big(\frac{3}{18}\Big) = 8$$

$$\sum_{\substack{\delta|N \\ 0 \quad g<\delta}} r_{\delta,g}\frac{N}{6\delta} = \frac{N}{6}\Big(\frac{2}{18}+\frac{4}{36}-\frac{4}{18}\Big) = 0.$$

Looking at $O(\tau)$ at the cusp $\frac{\lambda}{\mu\epsilon}$, we get the following orders:

$\epsilon = 1 :$ order 0

$\epsilon = 2 :$ order 2

$\epsilon = 3 :$ order 0

$\epsilon = 4 :$ order 0

$\epsilon = 6 :$ order 6

$\epsilon = 9 :$ order 4

$\epsilon = 12 :$ order 3

$\epsilon = 18 :$ order 2

$\epsilon = 36 :$ order 4

and since all of the above orders are nonnegative, $O(\tau) \in M_2(\Gamma_0(36))$.

We now construct a basis for $M_2(\Gamma_0(36))$. More specifically, since $M_2(\Gamma_0(N)) \subset M_2(\Gamma_0(36))$ for any $N|36$, we find bases for such $M_2(\Gamma_0(N))$:

$$M_2(\Gamma_0(2)) : H_2^{(2,1)}(\tau)$$

$$M_2(\Gamma_0(3)) : H_2^{(3,1)}(\tau)$$

$$M_2(\Gamma_0(4)) : H_2^{(4,1)}(\tau) = H_2^{(2,1)}(\tau), \quad H_2^{(4,2)}(\tau)$$

$$M_2(\Gamma_0(6)) : H_2^{(2,1)}(\tau), \quad H_2^{(3,1)}(\tau), \quad H_2^{(6,1)}(\tau)$$

$$M_2(\Gamma_0(9)) : H_2^{(3,1)}(\tau), \quad H_2^{(9,3)}(\tau), \quad \eta_{9,0}^{2}\eta_{9,3}^{-2}(\tau)$$

$$M_2(\Gamma_0(12)) : H_2^{(2,1)}(\tau), \quad H_2^{(3,1)}(\tau), \quad H_2^{(4,2)}(\tau), \quad H_2^{(6,1)}(\tau), \quad H_2^{(12,2)}(\tau)$$

$$\begin{aligned} M_2(\Gamma_0(18)) : & H_2^{(2,1)}(\tau), \quad H_2^{(3,1)}(\tau), \quad H_2^{(6,1)}(\tau), \quad H_2^{(9,3)}(\tau), \\ & H_2^{(18,3)}(\tau), \quad \eta_{9,0}^{2}\eta_{9,3}^{-2}(\tau), \quad \eta_{18,0}^{2}\eta_{18,6}^{-2}(\tau) \\ M_2(\Gamma_0(36)) : & H_2^{(2,1)}(\tau), \quad H_2^{(3,1)}(\tau), \quad H_2^{(4,2)}(\tau), \quad H_2^{(6,1)}(\tau), \\ & H_2^{(9,3)}(\tau), \quad H_2^{(12,2)}(\tau), \quad H_2^{(18,3)}(\tau), \quad H_2^{(36,6)}(\tau), \\ & \eta_{9,0}^{2}\eta_{9,3}^{-2}(\tau), \quad \eta_{18,0}^{2}\eta_{18,6}^{-2}(\tau), \quad \eta_{36,0}^{2}\eta_{36,12}^{-2}(\tau). \end{aligned}$$

(For the dimensions of $M_2(\Gamma_0(N))$, see [23], p. 23 – 25 and Theorem 1 of section 8 in [9]. To prove that $\eta_{9,0}^{2}\,\eta_{9,3}^{-2}(\tau) \in M_2(\Gamma_0(9))$, we use Theorem 1 and Theorem R:

$$\sum_{\substack{\delta | N \\ 0 \le g < \delta}} r_{\delta,g}\delta P_2\Big(\frac{g}{\delta}\Big) = 2 \cdot 9P_2(0) - 2 \cdot 9P_2\Big(\frac{3}{9}\Big) = 4$$

$$\sum_{\substack{\delta | N \\ 0 \le g < \delta}} r_{\delta,g}\frac{N}{6\delta} = \frac{9}{6}\Big(\frac{2}{9} - \frac{2}{9}\Big) = 0$$

and the order at $\frac{\lambda}{\mu\epsilon}$, where $\epsilon|9$ is:

$\epsilon = 1 :$ order 0

$\epsilon = 3 :$ order 0

$\epsilon = 9 :$ order 2.)

Now, the dimension of $M_2(\Gamma_0(36))$ is twelve, so we need one more modular form (linearly dependent from the eleven above). We could complete the basis using $O(\tau)$; to see that it is linearly independent from the eleven forms already listed, we look at the $q-$expansions and check their linear independence using Mathematica:

$$\begin{aligned}
H_2^{(2,1)}(\tau) &= 1 + 12\sum_{M=1}^{\infty} \sigma^{(2,1)}(M) q^M \\
&= 1 + 24q + 24q^2 + 96q^3 + 24q^4 + 144q^5 + 96q^6 + 192q^7 + 24q^8 \\
&\qquad + 312q^9 + 144q^{10} + 288q^{11} + 96q^{12} + \dots \\
H_2^{(3,1)}(\tau) &= 1 + 12\sum_{M=1}^{\infty} \sigma^{(3,1)}(M) q^M \\
&= 1 + 12q + 36q^2 + 12q^3 + 84q^4 + 72q^5 + 36q^6 + 96q^7 + 180q^8 \\
&\qquad + 12q^9 + 216q^{10} + 144q^{11} + 84q^{12} + \dots \\
H_2^{(4,2)}(\tau) &= 1 + 6\sum_{M=1}^{\infty} \sigma^{(4,2)}(M) q^M \\
&= 1 + 24q^2 + 24q^4 + 96q^6 + 24q^8 + 144q^{10} + 96q^{12} + \dots \\
H_2^{(6,1)}(\tau) &= 1 - 12\sum_{M=1}^{\infty} \sigma^{(6,1)}(M) q^M \\
&= 1 - 12q - 12q^2 - 12q^3 - 12q^4 - 72q^5 - 12q^6 - 96q^7 - 12q^8 \\
&\qquad - 12q^9 - 72q^{10} - 144q^{11} - 12q^{12} + \dots \\
H_2^{(9,3)}(\tau) &= 1 + 4\sum_{M=1}^{\infty} \sigma^{(9,3)}(M) q^M \\
&= 1 + 12q^3 + 36q^6 + 12q^9 + 84q^{12} + \dots \\
H_2^{(12,2)}(\tau) &= 1 - 6\sum_{M=1}^{\infty} \sigma^{(12,2)}(M) q^M \\
&= 1 - 12q^2 - 12q^4 - 12q^6 - 12q^8 - 72q^{10} - 12q^{12} - \dots \\
H_2^{(18,3)}(\tau) &= 1 - 4\sum_{M=1}^{\infty} \sigma^{(18,3)}(M) q^M \\
&= 1 - 12q^3 - 12q^6 - 12q^9 - 12q^{12} - \dots \\
H_2^{(36,6)}(\tau) &= 1 - 2\sum_{M=1}^{\infty} \sigma^{(36,6)}(M) q^M \\
&= 1 - 12q^6 - 12q^{12} - \dots \\
\eta_{9,0}^2\,\eta_{9,3}^{-2}(\tau) &= q^2 \prod_{n=1}^{\infty} \frac{(1-q^{9n})^6}{(1-q^{3n})^2} \\
&= q^2 + 2q^5 + 5q^8 + 4q^{11} + \dots
\end{aligned}$$

$$\begin{aligned}\eta_{18,0}^{2}\,\eta_{18,6}^{-2}(\tau) &= q^{4}\prod_{n=1}^{\infty}\frac{(1-q^{18n})^{6}}{(1-q^{6n})^{2}}\\ &= q^{4}+2q^{10}+\dots\\ \eta_{36,0}^{2}\,\eta_{36,12}^{-2}(\tau) &= q^{8}\prod_{n=1}^{\infty}\frac{(1-q^{36n})^{6}}{(1-q^{12n})^{2}}\\ &= q^{8}+\dots\\ O(\tau) &= \eta_{18,0}^{2}\,\eta_{36,6}^{4}\,\eta_{18,3}^{-4}(\tau)\\ &= q^{4}+4q^{7}+6q^{10}+\dots.\end{aligned}$$

To check linear independence, it suffices to consider the coefficients up to q^{12}; specifically, it suffices to consider the coefficients of the $q^0, q^1, q^2, \dots, q^{10}$, and q^{12} terms.

Notice that the nonzero coefficients of $\eta_{9,0}^{2}\,\eta_{9,3}^{-2}(\tau)$ only occur as coefficients of q^n where $n \equiv 2 \pmod 3$; for $\eta_{18,0}^{2}\,\eta_{18,6}^{-2}(\tau)$, they are non-zero only when $n \equiv 4 \pmod 6$; and for $\eta_{36,0}^{2}\,\eta_{36,12}^{-2}(\tau)$, they are non-zero only when $n \equiv 8 \pmod{12}$.

Now, if we write $P(\tau)$ in terms of this basis, we get (via Mathematica)

$$\begin{aligned}P(\tau) &= \sum_{M=0}^{\infty} p_4(M)q^{6M+1}\\ &= \frac{1}{12}H_2^{(3,1)}(\tau) - \frac{1}{12}H_2^{(9,3)}(\tau) - 3\eta_{9,0}^{2}\,\eta_{9,3}^{-2}(\tau) - 6\eta_{18,0}^{2}\,\eta_{18,6}^{-2}(\tau) - O(\tau).\end{aligned}$$

Reading off the coefficients of q^{6M+1}, we have

$$\begin{aligned}p_4(M) &= \frac{1}{12}12\sigma^{(3,1)}(6M+1) - \frac{1}{12}4\sigma^{(9,3)}(6M+1) - o_4(2M-1)\\ &= \sigma^{(3,1)}(6M+1) - o_4(2M-1)\\ &= \sigma(6M+1) - o_4(2M-1)\end{aligned}$$

(any divisor of $6M+1$ must be congruent to $\pm 1 \pmod 3$).

One can show that

$$M \equiv 1 (\text{mod } 2) \Rightarrow \sigma(6M+1) \equiv 0 \pmod 4$$

$$\text{and } M \equiv 1 (\text{mod } 4) \Rightarrow \sigma(6M+1) \equiv 0 \pmod 8;$$

from this, (9) and (10) follow:

$$M \equiv 1 (\text{mod } 2) \Rightarrow p_4(M) + o_4(2M-1) \equiv 0 \pmod 4$$

$$\text{and } M \equiv 1 (\text{mod } 4) \Rightarrow p_4(M) + o_4(2M-1) \equiv 0 \pmod 8.$$

If we read off the coefficients of q^{6M+4}, we get

$$0 = \sigma^{(3,1)}(6M+4) - \frac{1}{3}\sigma^{(9,3)}(6M+4) - 6a_{6M+4} - o_4(2M),$$

or

$$o_4(2M) = \sigma(6M+4) - 6a_{6M+4},$$

where a_m denotes the coefficient of q^m of $\eta_{18,0}^{2}\,\eta_{18,6}^{-2}(\tau)$. Since $a_m \in \mathsf{Z}$ for all m, we can use

$$M \equiv 1 (\text{mod } 2) \Rightarrow \sigma(6M+4) \equiv 0 \ (\text{mod } 18)$$

to derive (11):

$$M \equiv 1 (\text{mod } 2) \Rightarrow o_4(2M) \equiv 0 \ (\text{mod } 6).$$

To complete the basis for $M_2(\Gamma_0(36))$, we could instead use $\eta_{1,0}^{1/2}$. Replace τ by 6τ (q by q^6) in (17) and then raise both sides to the fourth power:

$$\eta_{6,0}^{2}(\tau) = q\Big(\sum_{n\in\mathsf{Z}}(-1)^n q^{9n^2-3n}\Big)^4 = \sum_{M=0}^{\infty} p_4^*(M) q^{6M+1},$$

where

$$p_4^*(M) = \sum_{M=\rho_1+\rho_2+\rho_3+\rho_4} (-1)^{\omega_1+\omega_2+\omega_3+\omega_4},$$

this last sum being taken over all of the ways of writing M as a sum of four pentagonal numbers $\rho_1, \rho_2, \rho_3, \rho_4$, with ρ_i denoting the ω_i^{th} pentagonal number (i.e. $\rho_i = \frac{3}{2}\omega_i^2 - \frac{1}{2}\omega_i$). Note that

$$-p_4(M) \ \square\ p_4^*(M) \ \square\ p_4(M)$$

for all M.

First, $\eta_{6,0}^{2}(\tau) \in M_2(\Gamma_0(36))$ by Theorem 1 and Theorem R:

$$\sum_{\substack{\delta \mid N \\ 0 \quad g<\delta}} r_{\delta,g}\,\delta P_2\Big(\frac{g}{\delta}\Big) = 2 \cdot 6P_2(0) = 2$$

$$\sum_{\substack{\delta \mid N \\ 0 \quad g<\delta}} r_{\delta,g}\frac{N}{6\delta} = 2\frac{36}{6\cdot 6} = 2;$$

and the order of $\eta^2_{6,0}(\tau)$ at the cusp $\frac{\lambda}{\mu\epsilon}$ is:

$\epsilon = 1:$ order 1

$\epsilon = 2:$ order 2

$\epsilon = 3:$ order 3

$\epsilon = 4:$ order 1

$\epsilon = 6:$ order 6

$\epsilon = 9:$ order 1

$\epsilon = 12:$ order 2

$\epsilon = 18:$ order 3

$\epsilon = 36:$ order 1.

Note that all of the orders are strictly positive, so that $\eta^2_{6,0}(\tau)$ is in fact a cusp-form on $\Gamma_0(36)$.

Using the first few terms of $\eta^2_{6,0}(\tau)$,

$$\eta^2_{6,0}(\tau) = q - 4q^7 + \dots,$$

we can write $P(\tau)$ (again using Mathematica) as

$$\begin{aligned} P(\tau) = {} & \frac{1}{12}H_2^{(3,1)}(\tau) + \frac{1}{36}H_2^{(6,1)}(\tau) - \frac{1}{12}H_2^{(9,3)}(\tau) \\ & + \frac{1}{18}H_2^{(12,2)}(\tau) - \frac{1}{36}H_2^{(18,3)}(\tau) - \frac{1}{18}H_2^{(36,6)}(\tau) - 2\eta^2_{9,0}\,\eta^{-2}_{9,3}(\tau) \\ & - 6\eta^2_{18,0}\,\eta^{-2}_{18,6}(\tau) - 4\eta^2_{36,0}\,\eta^{-2}_{36,12}(\tau) + \frac{1}{3}\eta^2_{6,0}(\tau). \end{aligned}$$

Reading off the coefficients of q^{6M+1}, we find

$$\begin{aligned} p_4(M) &= \frac{1}{12}12\sigma^{(3,1)}(6M+1) + \frac{1}{36}(-12\sigma^{(6,1)}(6M+1)) - \frac{1}{12}4\sigma^{(9,3)}(6M+1) \\ &\quad + \frac{1}{18}(-6\sigma^{(12,2)}(6M+1)) - \frac{1}{36}(-4\sigma^{(18,3)}(6M+1)) \\ &\quad - \frac{1}{18}(-2\sigma^{(36,6)}(6M+1)) + \frac{1}{3}p_4^*(M) \\ &= \sigma^{(3,1)}(6M+1) - \frac{1}{3}\sigma^{(6,1)}(6M+1) + \frac{1}{3}p_4^*(M) \\ &= \frac{2}{3}\sigma(6M+1) + \frac{1}{3}p_4^*(M). \end{aligned}$$

From $p_4^*(M) \geq -p_4(M)$, we get

$$\begin{aligned} p_4(M) &= \frac{2}{3}\sigma(6M+1) + \frac{1}{3}p_4^*(M) \\ &\geq \frac{2}{3}\sigma(6M+1) - \frac{1}{3}p_4(M) \end{aligned}$$

which implies $p_4(M) \geq \frac{1}{2}\sigma(6M+1)$. Conversely, from $p_4^*(M) \square p_4(M)$, we get

$$\begin{aligned} p_4(M) &= \frac{2}{3}\sigma(6M+1) + \frac{1}{3}p_4^*(M) \\ &\square \frac{2}{3}\sigma(6m+1) + \frac{1}{3}p_4(M) \end{aligned}$$

which implies $p_4(M) \square \sigma(6M+1)$. This establishes (12):

$$\frac{1}{2}\sigma(6M+1) \square p_4(M) \square \sigma(6M+1).$$

We could also try to use this basis to find bounds for $o_4(M)$. First, writing $O(\tau)$ in terms of this basis, we have (using Mathematica)

$$\begin{aligned} O(\tau) = &-\frac{1}{36}H_2^{(6,1)}(\tau) - \frac{1}{18}H_2^{(12,2)}(\tau) + \frac{1}{36}H_2^{(18,3)}(\tau) \\ &+ \frac{1}{18}H_2^{(36,6)}(\tau) - \eta_{9,0}^2\,\eta_{9,3}^{-2}(\tau) + 4\eta_{36,0}^2\,\eta_{36,12}^{-2}(\tau) - \frac{1}{3}\eta_{6,0}^2(\tau). \end{aligned}$$

Reading off the coefficients of q^{3M+4} yields

$$\begin{aligned} o_4(M) = &-\frac{1}{36}(-12\sigma^{(6,1)}(3M+4)) - \frac{1}{18}(-6\sigma^{(12,2)}(3M+4)) \\ &+ \frac{1}{36}(-4\sigma^{(18,3)}(3M+4)) + \frac{1}{18}(-2\sigma^{(36,6)}(3M+4)) \\ &- \frac{1}{3}p_4^*\Big(\frac{M+1}{2}\Big) \\ = &\ \frac{1}{3}\sigma^{(6,1)}(3M+4) + \frac{1}{3}\sigma^{(12,2)}(3M+4) - \frac{1}{3}p_4^*\Big(\frac{M+1}{2}\Big), \end{aligned}$$

where it is understood that $p_4^*\Big(\frac{M+1}{2}\Big) = 0$ if $M+1$ is odd.

If M is odd, then $\sigma^{(6,1)}(3M+4) = \sigma(3M+4)$ and $\sigma^{(12,2)}(3M+4) = 0$, so

$$M \equiv 1 (\text{mod } 2) \qquad \Rightarrow \qquad o_4(M) = \frac{1}{3}\sigma(3M+4) - \frac{1}{3}p_4^*\Big(\frac{M+1}{2}\Big).$$

If $M \equiv 2 \pmod 4$, then $M+1$ is odd, so $p_4^*\Big(\frac{M+1}{2}\Big) = 0$. Let $M = 2(2K+1)$; then

$$\sigma^{(6,1)}(3M+4) = \sigma^{(6,1)}(2(6K+5)) = \sigma^{(6,1)}(6K+5) = \sigma(6K+5)$$

and

$$\sigma^{(12,2)}(3M+4) = \sigma^{(12,2)}(12K+10) = 2\sigma^{(6,1)}(6K+5) = 2\sigma(6K+5).$$

So

$$\begin{aligned} o_4(M) &= \frac{1}{3}\sigma^{(6,1)}(3M+4) + \frac{1}{3}\sigma^{(12,2)}(3M+4) \\ &= \frac{1}{3}\sigma(6K+5) + \frac{2}{3}\sigma(6K+5) \\ &= \sigma(6K+5) \\ &= \frac{\sigma(2)\sigma(6K+5)}{\sigma(2)} \\ &= \frac{1}{3}\sigma(2(6K+5)) \\ &= \frac{1}{3}\sigma(3M+4), \end{aligned}$$

which is (13).

If $M \equiv 0 \pmod 4$, then $p_4^*\left(\frac{M+1}{2}\right) = 0$ again. Let $M = 4K$; then

$$\sigma^{(6,1)}(3M+4) = \sigma^{(6,1)}(12K+4) = \sigma^{(6,1)}(4(3K+1)) = \sigma^{(6,1)}(3K+1)$$

and

$$\begin{aligned} \sigma^{(12,2)}(3M+4) &= \sigma^{(12,2)}(12K+4) = \sigma^{(12,2)}(4(3K+1)) \\ &= 2\sigma^{(6,1)}(2(3K+1)) = 2\sigma^{(6,1)}(3K+1), \end{aligned}$$

so

$$\begin{aligned} o_4(M) &= \frac{1}{3}\sigma^{(6,1)}(3M+4) + \frac{1}{3}\sigma^{(12,2)}(3M+4) \\ &= \frac{1}{3}\sigma^{(6,1)}(3K+1) + \frac{1}{3}2\sigma^{(6,1)}(3K+1) \\ &= \sigma^{(6,1)}(3K+1), \end{aligned}$$

which is (14).

If k is an integer, relatively prime to 3, then $\sigma^{(6,1)}(k) = \frac{1}{3}\sigma(4k) - \frac{4}{3}\sigma(k)$, so

$$\begin{aligned} o_4(M) &= \frac{1}{3}\sigma(4(3K+1)) - \frac{4}{3}\sigma(3K+1) \\ &= \frac{1}{3}\sigma(3M+4) - \frac{4}{3}\sigma\Big(\frac{3M+4}{4}\Big). \end{aligned}$$

In the case of $M \equiv 1 \pmod 2$, we could try to use the inequality $p_4^*(M)$ □ $p_4(M)$ □ $\sigma(6M+1)$ to find a lower bound for $o_4(M)$, but the best we can do is zero:

$$\begin{aligned} o_4(M) &= \frac{1}{3}\sigma(3M+4) - \frac{1}{3}p_4^*\Big(\frac{M+1}{2}\Big) \\ &\geq \frac{1}{3}\sigma(3M+4) - \frac{1}{3}\sigma(6\Big(\frac{M+1}{2}\Big)+1) \\ &= 0. \end{aligned}$$

So we could conceivably have $o_4(M) = 0$ for M odd. If M is even, then we have $o_4(M) > 0$ from the above formulas; however, the best lower bound we can find will be 1. Indeed, $o_4(M)$ is equal to 1 for infinitely many values of M: if k is positive and even, then for $M = \frac{4}{3}(2^k - 1)$, we find that $M = 4\,\frac{2^k-1}{3} \equiv 0 \pmod 4$, and hence

$$\begin{aligned} o_4(M) &= \frac{1}{3}\sigma(3M+4) - \frac{4}{3}\sigma\Big(\frac{3M+4}{4}\Big) \\ &= \frac{1}{3}\sigma(2^{k+2}) - \frac{4}{3}\sigma(2^k) \\ &= \frac{2^{k+3} - 1 - 4(2^{k+1} - 1)}{3} \\ &= 1. \end{aligned}$$

Some examples of this are the following:

$$\begin{aligned} 4 &= 1+1+1+1 \\ 20 &= 5+5+5+5 \\ 84 &= 21+21+21+21 \\ 340 &= 85+85+85+85 \\ 1364 &= 341+341+341+341. \end{aligned}$$