

II. The Generalized Dedekind Eta Function

We now introduce the generalized Dedekind eta function $\eta_{\delta,g}(\tau)$. Let \mathfrak{h} denote the upper half plane $\{a + bi \mid b > 0\}$. For $g, \delta \in \mathbb{Z}$, with $\delta > 0$, we define

$$\eta_{\delta,g}(\tau) := e^{\pi i \tau \delta P_2(g/\delta)} \prod_{\substack{m \equiv g \pmod{\delta} \\ m > 0}} (1 - q^m) \prod_{\substack{m \equiv -g \pmod{\delta} \\ m > 0}} (1 - q^m)$$

(where $q = e^{2\pi i \tau}$, $\tau \in \mathfrak{h}$). Note that if $g \equiv g' \pmod{\delta}$, then $P_2(g/\delta) = P_2(g'/\delta)$, so we may assume $0 \leq g < \delta$. If $g = 0$ and $\delta = 1$, then we get the square of the original eta function:

$$\eta_{1,0}(\tau) = \eta^2(\tau),$$

where $\eta(\tau) = e^{2\pi i \tau / 24} \prod_{m=1}^{\infty} (1 - q^m)$. Some of the properties of these functions that are easy to verify include the following:

$$\eta_{\delta,0}(\tau) = \eta^2(\delta\tau)$$

$$\eta_{\delta, \frac{\delta}{2}}(\tau) = \eta_{\frac{\delta}{2},0}(\tau) / \eta_{\delta,0}(\tau) \quad (\text{if } \delta \text{ is even})$$

$$\eta_{\delta,g}(\tau) = \eta_{\delta,\delta-g}(\tau)$$

$$\text{for } r \in \mathbb{N}, \quad \eta_{\delta,g}(r\tau) = \eta_{\deltar,gr}(\tau). \quad (4)$$

These eta functions are a variation of those defined by Schoeneberg in [22]. If $\eta_{\left(\frac{g}{\delta}\right)}^{(\text{Sch})}(\tau)$ denotes Schoeneberg's version, then

$$\eta_{\delta,g}(\tau) = \eta_{\left(\frac{g}{\delta}\right)}^{(\text{Sch})}(\delta\tau).$$

Since Schoeneberg's version of the eta function transforms on $\Gamma = \text{SL}_2(\mathbb{Z})$, $\eta_{\delta,g}(\tau)$ should transform on $\Gamma_0(\delta) = \left\{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \mid c \equiv 0 \pmod{\delta} \right\}$. In particular, equation (30) of chapter 8 in [22] becomes

$$\eta_{\delta,g}(A\tau) = \eta_{\delta,ag}(\tau) \cdot \exp\left(\text{sgn } c \left\langle \text{sgn } c \cdot \log(c\tau + d) - \frac{\pi i}{2} \right\rangle\right) \nu_{\delta,g}(A)$$

where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(\delta)$,

$$\langle x \rangle = \begin{cases} x, & \text{if } g \equiv 0 \pmod{\delta} \\ 0, & \text{if } g \not\equiv 0 \pmod{\delta}, \end{cases}$$

and $\nu_{\delta,g}$ is a character given by

$$\nu_{\delta,g}(A) = \begin{cases} \exp(\pi i [\frac{a}{c} \delta P_2(\frac{g}{\delta}) + \frac{d}{c} \delta P_2(\frac{ag}{\delta}) - 2 \operatorname{sgn} c s(a, c/\delta; 0, g/\delta)]), & \text{if } c \neq 0 \\ \exp(\pi i \frac{b}{d} \delta P_2(\frac{g}{\delta})), & \text{if } c = 0. \end{cases}$$

As in [22], the products defining these eta functions converge uniformly on compact subsets of \mathfrak{h} , and $\eta_{\delta,g}(\tau)$ is never zero on \mathfrak{h} .

If $A \in \Gamma_1(\delta) = \left\{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \mid c \equiv 0 \pmod{\delta} \text{ and } a \equiv b \equiv 1 \pmod{\delta} \right\}$, then $a \equiv 1 \pmod{\delta}$, so $\eta_{\delta,g}(\tau)$ is a modular function of weight zero [resp., one] if $0 < g < \delta$ [resp., $g = 0$] on $\Gamma_1(\delta)$ with multiplier $\nu_{\delta,g}$ [resp., $e^{-\pi i \operatorname{sgn} c} \nu_{\delta,0}$]. However, if $A \in \Gamma_0(\delta)$, then we may not get the same eta function back, e.g.

$$\eta_{5,1}\left(\frac{2\tau+1}{5\tau+3}\right) = e^{-4\pi i/15} \eta_{5,2}(\tau).$$

We now prove a result similar to one of Robins. In [21], Robins proved the following:

Proposition. Let $N \in \mathbb{N}$,

$$f(\tau) = \prod_{\substack{\delta \mid N \\ 0 \leq g < \delta}} \eta_{\delta,g}^{r_{\delta,g}}(\tau)$$

where $r_{\delta,g} \in \mathbb{Z}$, with $r_{\delta,0}$ and $r_{\delta,\frac{\delta}{2}}$ allowed to be half-integers, and let $k = \sum_{\delta \mid N} r_{\delta,0} \in \mathbb{Z}$.

If

$$\sum_{\substack{\delta \mid N \\ 0 \leq g < \delta}} r_{\delta,g} \delta P_2\left(\frac{g}{\delta}\right) \equiv 0 \pmod{2}$$

and

$$\sum_{\substack{\delta \mid N \\ 0 \leq g < \delta}} r_{\delta,g} \frac{N}{\delta} \frac{1}{6} \equiv 0 \pmod{2},$$

then f is a modular function of weight k on $\Gamma_1(N)$ (with trivial multiplier).

We now prove a similar theorem for $\Gamma_0(N)$:

Theorem 1. Let $N \in \mathbb{N}$,

$$f(\tau) = \prod_{\substack{\delta|N \\ 0 < g < \delta}} \eta_{\delta,g}^{r_{\delta,g}}(\tau)$$

where $r_{\delta,g} \in \mathbb{Z}$, $r_{\delta,ag} = r_{\delta,g}$ for all a with $(a, N) = 1$, and $k = \sum_{\delta|N} r_{\delta,0}$ be an even integer.

If

$$\sum_{\substack{\delta|N \\ 0 < g < \delta}} r_{\delta,g} \delta P_2\left(\frac{g}{\delta}\right) \equiv 0 \pmod{2} \quad (5)$$

and

$$\sum_{\substack{\delta|N \\ 0 < g < \delta}} r_{\delta,g} \frac{N}{\delta} \frac{1}{6} \equiv 0 \pmod{2} \quad (6)$$

then f is modular of weight k on $\Gamma_0(N)$ (with trivial multiplier).

Proof. As in [16], it suffices to consider those A with $(a, 6) = 1$ and $a > 0$, $c > 0$ (since these matrices generate $\Gamma_0(N)$). For the purposes of making this proof easier, we will also assume that $r_{\delta, \frac{\delta}{2}} = 0$ for all even δ . We lose no generality with this assumption since $\eta_{\delta, \frac{\delta}{2}}(\tau) = \eta_{\frac{\delta}{2}, 0}(\tau) / \eta_{\delta, 0}(\tau)$ and the congruences (5) and (6) do not change ($r\delta P_2(1/2) = r\frac{\delta}{2} \frac{1}{6} - r\delta \frac{1}{6}$ and $r\frac{N}{6\delta} = r\frac{N}{6\delta/2} - r\frac{N}{6\delta}$). Now, f transforms as

$$\begin{aligned} f(A\tau) &= \prod_{\substack{\delta|N \\ 0 < g < \delta}} \eta_{\delta, ag}^{r_{\delta,g}}(\tau) \exp\left(r_{\delta,g} \left\langle \log(c\tau + d) - \frac{\pi i}{2} \right\rangle\right) \nu_{\delta,g}^{r_{\delta,g}}(A) \\ &= f(\tau) \prod_{\substack{\delta|N \\ (g=0)}} \exp\left(r_{\delta,0} \left(\log(c\tau + d) - \frac{\pi i}{2} \right)\right) \prod_{\substack{\delta|N \\ 0 < g < \delta}} \nu_{\delta,g}^{r_{\delta,g}}(A) \\ &= f(\tau) (c\tau + d)^k e^{-\pi i k / 2} \prod_{\substack{\delta|N \\ 0 < g < \delta}} \nu_{\delta,g}^{r_{\delta,g}}(A). \end{aligned}$$

We want to show that

$$e^{-\pi i k / 2} \prod_{\substack{\delta|N \\ 0 < g < \delta}} \nu_{\delta,g}^{r_{\delta,g}}(A) = 1.$$

This multiplier is

$$\begin{aligned}
& \prod_{\substack{\delta|N \\ (g=0)}} e^{-\pi i r_{\delta,0}/2} \prod_{\substack{\delta|N \\ 0 < g < \delta}} \nu_{\delta,g}^{r_{\delta,g}}(A) \\
&= \prod_{\substack{\delta|N \\ (g=0)}} e^{-\pi i r_{\delta,0}/2} \prod_{\substack{\delta|N \\ 0 < g < \delta}} \exp\left(r_{\delta,g} \pi i \left[\frac{a}{c} \delta P_2\left(\frac{g}{\delta}\right) + \frac{d}{c} \delta P_2\left(\frac{ag}{\delta}\right) - 2s\left(a, \frac{c}{\delta}; 0, \frac{g}{\delta}\right) \right]\right) \\
&= \exp\left(\pi i \left[\sum_{\substack{\delta|N \\ (g=0)}} \frac{-r_{\delta,0}}{2} + \sum_{\substack{\delta|N \\ (g=0)}} r_{\delta,0} \left(\frac{a+d}{6c} \delta - 2s\left(a, \frac{c}{\delta}\right) \right) \right. \right. \\
&\quad \left. \left. + \sum_{\substack{\delta|N \\ 0 < g < \delta}} r_{\delta,g} \left(\frac{a+d}{c} \delta P_2\left(\frac{g}{\delta}\right) - 2s\left(a, \frac{c}{\delta}; 0, \frac{g}{\delta}\right) \right) \right] \right),
\end{aligned}$$

so we want to show

$$\begin{aligned}
& \sum_{\substack{\delta|N \\ (g=0)}} r_{\delta,0} \left(-\frac{1}{2} + \frac{a+d}{6c} \delta - 2s\left(a, \frac{c}{\delta}\right) \right) \\
&+ \sum_{\substack{\delta|N \\ 0 < g < \delta}} r_{\delta,g} \left(\frac{a+d}{c} \delta P_2\left(\frac{g}{\delta}\right) - 2s\left(a, \frac{c}{\delta}; 0, \frac{g}{\delta}\right) \right) \equiv 0 \pmod{2}.
\end{aligned}$$

Using the reciprocity of Meyer sums, we get

$$\begin{aligned}
& \sum_{\substack{\delta|N \\ (g=0)}} r_{\delta,0} \left(-\frac{1}{2} + \frac{a+d}{6c} \delta - 2s\left(a, \frac{c}{\delta}\right) \right) \\
&\quad + \sum_{\substack{\delta|N \\ 0 < g < \delta}} r_{\delta,g} \left(\frac{a+d}{c} \delta P_2\left(\frac{g}{\delta}\right) - 2s\left(a, \frac{c}{\delta}; 0, \frac{g}{\delta}\right) \right) \\
&= \sum_{\substack{\delta|N \\ (g=0)}} \frac{-r_{\delta,0}}{2} + \sum_{\substack{\delta|N \\ (g=0)}} r_{\delta,0} \frac{a+d}{6c} \delta + \sum_{\substack{\delta|N \\ 0 < g < \delta}} r_{\delta,g} \frac{a+d}{c} \delta P_2\left(\frac{g}{\delta}\right) \\
&\quad + \sum_{\substack{\delta|N \\ (g=0)}} \left(-2r_{\delta,0} \left[-s\left(\frac{c}{\delta}, a\right) - \frac{1}{4} + \frac{1}{12} \frac{a\delta}{c} + \frac{\delta}{12ac} + \frac{c}{12a\delta} \right] \right) \\
&\quad + \sum_{\substack{\delta|N \\ 0 < g < \delta}} \left(-2r_{\delta,g} \left[-s\left(\frac{c}{\delta}, a; \frac{g}{\delta}, 0\right) + \frac{1}{2} \frac{a\delta}{c} P_2\left(\frac{g}{\delta}\right) \right. \right. \\
&\quad \left. \left. + \frac{\delta}{2ac} P_2\left(\frac{ag}{\delta}\right) + \frac{c}{12a\delta} \right] \right)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{\substack{\delta|N \\ (g=0)}} \frac{-r_{\delta,0}}{2} + \sum_{\substack{\delta|N \\ (g=0)}} r_{\delta,0} \frac{a+d}{6c} \delta + \sum_{\substack{\delta|N \\ 0 < g < \delta}} r_{\delta,g} \frac{a+d}{c} \delta P_2\left(\frac{g}{\delta}\right) \\
&\quad + \sum_{\substack{\delta|N \\ (g=0)}} 2r_{\delta,0} s\left(\frac{c}{\delta}, a\right) + \frac{1}{2} \sum_{\substack{\delta|N \\ (g=0)}} r_{\delta,0} - \sum_{\substack{\delta|N \\ (g=0)}} \frac{r_{\delta,0} a \delta}{6c} \\
&\quad - \sum_{\substack{\delta|N \\ (g=0)}} \frac{r_{\delta,0} \delta}{6ac} - \sum_{\substack{\delta|N \\ (g=0)}} r_{\delta,0} \frac{c}{6a\delta} + \sum_{\substack{\delta|N \\ 0 < g < \delta}} 2r_{\delta,g} s\left(\frac{c}{\delta}, a; \frac{g}{\delta}, 0\right) \\
&\quad - \sum_{\substack{\delta|N \\ 0 < g < \delta}} r_{\delta,g} \frac{a\delta}{c} P_2\left(\frac{g}{\delta}\right) - \sum_{\substack{\delta|N \\ 0 < g < \delta}} r_{\delta,g} \frac{\delta}{ac} P_2\left(\frac{ag}{\delta}\right) - \sum_{\substack{\delta|N \\ 0 < g < \delta}} r_{\delta,g} \frac{c}{6a\delta} \\
&= \sum_{\substack{\delta|N \\ (g=0)}} r_{\delta,0} \left(\frac{d\delta}{6c} - \frac{\delta}{6ac} - \frac{c}{6a\delta} + 2s\left(\frac{c}{\delta}, a\right) \right) \\
&\quad + \sum_{\substack{\delta|N \\ 0 < g < \delta}} r_{\delta,g} \left(\frac{d\delta}{c} P_2\left(\frac{g}{\delta}\right) - \frac{\delta}{ac} P_2\left(\frac{g}{\delta}\right) - \frac{c}{6a\delta} + 2s\left(\frac{c}{\delta}, a; \frac{g}{\delta}, 0\right) \right) \\
&= \sum_{\substack{\delta|N \\ (g=0)}} r_{\delta,0} \left(\frac{\delta}{6c} \left(\frac{ad-1}{a} \right) - \frac{c}{6a\delta} + 2s\left(\frac{c}{\delta}, a\right) \right) \\
&\quad + \sum_{\substack{\delta|N \\ 0 < g < \delta}} r_{\delta,g} \left(\frac{\delta}{c} P_2\left(\frac{g}{\delta}\right) \left(\frac{ad-1}{a} \right) - \frac{c}{6a\delta} + 2s\left(\frac{c}{\delta}, a; \frac{g}{\delta}, 0\right) \right) \\
&= \sum_{\substack{\delta|N \\ (g=0)}} r_{\delta,0} \left(\frac{b\delta}{6a} - \frac{c}{6a\delta} + 2s\left(\frac{c}{\delta}, a\right) \right) \\
&\quad + \sum_{\substack{\delta|N \\ 0 < g < \delta}} r_{\delta,g} \left(\frac{b\delta}{a} P_2\left(\frac{g}{\delta}\right) - \frac{c}{6a\delta} + 2s\left(\frac{c}{\delta}, a; \frac{g}{\delta}, 0\right) \right) \\
&= \sum_{\substack{\delta|N \\ 0 \square g < \delta}} r_{\delta,g} \left(\frac{b}{a} \delta P_2\left(\frac{g}{\delta}\right) - \frac{c}{6a\delta} + 2s\left(\frac{c}{\delta}, a; \frac{g}{\delta}, 0\right) \right).
\end{aligned}$$

So it suffices to show

$$\sum_{\substack{\delta|N \\ 0 \square g < \delta}} r_{\delta,g} \left(\frac{b}{a} \delta P_2\left(\frac{g}{\delta}\right) - \frac{c}{6a\delta} + 2s\left(\frac{c}{\delta}, a; \frac{g}{\delta}, 0\right) \right) \equiv 0 \pmod{2}.$$

Note that this is equivalent to showing that

$$\sum_{\substack{\delta|N \\ 0 \square g < \delta}} r_{\delta,g} \left[b\delta P_2\left(\frac{g}{\delta}\right) - \frac{c}{6\delta} + 2as\left(\frac{c}{\delta}, a; \frac{g}{\delta}, 0\right) \right] \equiv 0 \pmod{2a}.$$

By (5) and (6) we know that $\sum r_{\delta,g} \delta P_2\left(\frac{g}{\delta}\right)$ and $\sum r_{\delta,g} \frac{N}{6\delta}$ are even integers; since $N|c$, it suffices to show

$$\sum_{\substack{\delta|N \\ 0 < g < \delta}} 2r_{\delta,g} a s\left(\frac{c}{\delta}, a; \frac{g}{\delta}, 0\right) \equiv 0 \pmod{2} \quad (7)$$

and

$$\sum_{\substack{\delta|N \\ 0 < g < \delta}} r_{\delta,g} \left[b\delta P_2\left(\frac{g}{\delta}\right) - \frac{c}{6\delta} + 2a s\left(\frac{c}{\delta}, a; \frac{g}{\delta}, 0\right) \right] \equiv 0 \pmod{a} \quad (8)$$

(recall that $(a, 2) = 1$).

For (7), note that

$$\begin{aligned} 2a s\left(\frac{c}{\delta}, a; \frac{g}{\delta}, 0\right) &= 2a \sum_{\mu \pmod{a}} \left(\left(\frac{\mu}{a} \right) \right) \left(\left(\frac{c\mu}{a\delta} + \frac{g}{\delta} \right) \right) \\ &= 2a \sum_{\mu=1}^{a-1} \left(\frac{\mu}{a} - \frac{1}{2} \right) \left(\left(\frac{c\mu}{a\delta} + \frac{g}{\delta} \right) \right) \\ &= \sum_{\mu=1}^{a-1} 2\mu \left(\left(\frac{c\mu}{a\delta} + \frac{g}{\delta} \right) \right) - a \sum_{\mu=1}^{a-1} \left(\left(\frac{c\mu}{a\delta} + \frac{g}{\delta} \right) \right). \end{aligned}$$

From the definition of $((x))$ and the fact that $\frac{c\mu}{a\delta} + \frac{g}{\delta} \notin \mathbb{Z}$, this last sum is

$$\begin{aligned} &2 \sum_{\mu=1}^{a-1} \left(\mu \left(\frac{c\mu}{a\delta} + \frac{g}{\delta} \right) - \mu \left\lfloor \frac{c\mu}{a\delta} + \frac{g}{\delta} \right\rfloor - \frac{1}{2}\mu \right) - a \sum_{\mu=1}^{a-1} \left(\left(\frac{c\mu}{a\delta} + \frac{g}{\delta} \right) \right) \\ &= 2 \sum_{\mu=1}^{a-1} \left(\frac{c\mu^2}{a\delta} + \frac{\mu g}{\delta} - \mu \left\lfloor \frac{c\mu}{a\delta} + \frac{g}{\delta} \right\rfloor - \frac{1}{2}\mu \right) - a \sum_{\mu=1}^{a-1} \left(\left(\frac{\mu}{a} + \frac{g}{\delta} \right) \right) \\ &= 2 \sum_{\mu=1}^{a-1} \left(\frac{c\mu^2}{a\delta} + \frac{\mu g}{\delta} - \mu \left\lfloor \frac{c\mu}{a\delta} + \frac{g}{\delta} \right\rfloor - \frac{1}{2}\mu \right) - a \sum_{\mu=1}^{a-1} \left(\frac{\mu}{a} + \frac{g}{\delta} - \left\lfloor \frac{\mu}{a} + \frac{g}{\delta} \right\rfloor - \frac{1}{2} \right). \end{aligned}$$

Since $\sum_{\mu=1}^{a-1} \mu = \frac{(a-1)a}{2}$ and $\sum_{\mu=1}^{a-1} \mu^2 = \frac{(a-1)a(2a-1)}{6}$, we find that

$$\begin{aligned}
& 2 \sum_{\mu=1}^{a-1} \left(\frac{c\mu^2}{a\delta} + \frac{\mu g}{\delta} - \mu \left\lfloor \frac{c\mu}{a\delta} + \frac{g}{\delta} \right\rfloor - \frac{1}{2} \mu \right) - a \sum_{\mu=1}^{a-1} \left(\frac{\mu}{a} + \frac{g}{\delta} - \left\lfloor \frac{\mu}{a} + \frac{g}{\delta} \right\rfloor - \frac{1}{2} \right) \\
&= 2 \left(\frac{c}{a\delta} \frac{(a-1)a(2a-1)}{6} + \frac{g}{\delta} \frac{a(a-1)}{2} - \frac{1}{2} \frac{a(a-1)}{2} - \sum_{\mu=1}^{a-1} \mu \left\lfloor \frac{c\mu}{a\delta} + \frac{g}{\delta} \right\rfloor \right) \\
&\quad - a \left(\frac{1}{a} \frac{a(a-1)}{2} + \frac{(a-1)g}{\delta} - \frac{1}{2}(a-1) - \sum_{\mu=1}^{a-1} \left\lfloor \frac{\mu}{a} + \frac{g}{\delta} \right\rfloor \right) \\
&= \frac{c}{\delta} \frac{(a-1)(2a-1)}{3} + \frac{g}{\delta} a(a-1) - \frac{1}{2} a(a-1) - 2 \sum_{\mu=1}^{a-1} \mu \left\lfloor \frac{c\mu}{a\delta} + \frac{g}{\delta} \right\rfloor \\
&\quad - \frac{1}{2} a(a-1) - a(a-1) \frac{g}{\delta} + \frac{1}{2} a(a-1) + a \sum_{\mu=1}^{a-1} \left\lfloor \frac{\mu}{a} + \frac{g}{\delta} \right\rfloor \\
&= \frac{c}{\delta} \frac{(a-1)(2a-1)}{3} - \frac{1}{2} a(a-1) - 2 \sum_{\mu=1}^{a-1} \mu \left\lfloor \frac{c\mu}{a\delta} + \frac{g}{\delta} \right\rfloor + a \sum_{\mu=1}^{a-1} \left\lfloor \frac{\mu}{a} + \frac{g}{\delta} \right\rfloor.
\end{aligned}$$

Now, $\delta|N$ and $N|c$, so $\frac{c}{\delta} \in \mathbb{Z}$. Also, since $(a, 6) = 1$, $\frac{(a-1)(2a-1)}{3}$ is even; and $2 \sum_{\mu=1}^{a-1} \mu \left\lfloor \frac{c\mu}{a\delta} + \frac{g}{\delta} \right\rfloor$ is certainly even. Therefore, reducing the last sum mod 2, we get

$$\begin{aligned}
& -\frac{1}{2} a(a-1) + a \sum_{\mu=1}^{a-1} \left\lfloor \frac{\mu}{a} + \frac{g}{\delta} \right\rfloor \\
& \equiv \left\lfloor \frac{ag}{\delta} \right\rfloor - \frac{a-1}{2} \pmod{2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \sum_{\substack{\delta|N \\ 0 < g < \delta}} 2r_{\delta,g} a s \left(\frac{c}{\delta}, a; \frac{g}{\delta}, 0 \right) \equiv \sum_{\substack{\delta|N \\ 0 < g < \delta}} r_{\delta,g} \left(\left\lfloor \frac{ag}{\delta} \right\rfloor - \frac{a-1}{2} \right) \\
&= \sum_{\delta|N} \sum_{d|\delta} \sum_{\substack{0 < g < \delta \\ (g, \delta) = d}} r_{\delta,g} \left(\left\lfloor \frac{ag}{\delta} \right\rfloor - \frac{a-1}{2} \right) \\
&= \sum_{\delta|N} \sum_{\substack{d|\delta \\ d < \delta}} \sum_{\substack{(g, \delta) = d \\ g < \delta}} \left(r_{\delta,g} \left\lfloor \frac{ag/d}{\delta/d} \right\rfloor - r_{\delta,g} \frac{a-1}{2} \right) - \sum_{\substack{\delta|N \\ (g=0)}} r_{\delta,0} \frac{a-1}{2}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{\delta|N} \sum_{\substack{d|\delta \\ d < \delta}} \left(r_{\delta,d} \left(\sum_{\left(\frac{g}{d}, \frac{\delta}{d}\right)=1} \left\lfloor \frac{ag/d}{\delta/d} \right\rfloor \right) - r_{\delta,d} \phi\left(\frac{\delta}{d}\right) \frac{a-1}{2} \right) - k \frac{a-1}{2} \\
&= \sum_{\delta|N} \sum_{\substack{d|\delta \\ d < \delta}} \left(r_{\delta,d} \phi\left(\frac{\delta}{d}\right) \frac{a-1}{2} - r_{\delta,d} \phi\left(\frac{\delta}{d}\right) \frac{a-1}{2} \right) - k \frac{a-1}{2} \\
&\hspace{15em} \text{(by Corollary 3 of Lemma 1)} \\
&= -k \frac{a-1}{2} \\
&\equiv 0 \pmod{2}.
\end{aligned}$$

It remains to prove (8), i. e. that

$$\sum_{\substack{\delta|N \\ 0 < g < \delta}} r_{\delta,g} \left[b\delta P_2\left(\frac{g}{\delta}\right) - \frac{c}{6\delta} + 2a s\left(\frac{c}{\delta}, a; \frac{g}{\delta}, 0\right) \right],$$

which we now know is an even integer, is in fact an integer congruent to 0 modulo a .

Now,

$$\begin{aligned}
&\sum_{\substack{\delta|N \\ 0 < g < \delta}} r_{\delta,g} \left[b\delta P_2\left(\frac{g}{\delta}\right) - \frac{c}{6\delta} + 2a s\left(\frac{c}{\delta}, a; \frac{g}{\delta}, 0\right) \right] \\
&= \sum_{\substack{\delta|N \\ (g=0)}} r_{\delta,0} \left[\frac{b\delta}{6} - \frac{c}{6\delta} + 2a s\left(\frac{c}{\delta}, a\right) \right] + \sum_{\substack{\delta|N \\ 0 < g < \delta}} \sum_{d|\delta} r_{\delta,d} \left[b\delta \left(\sum_{(g,\delta)=d} P_2\left(\frac{g}{\delta}\right) \right) \right. \\
&\hspace{15em} \left. - \phi\left(\frac{\delta}{d}\right) \frac{c}{6\delta} + \left(\sum_{(g,\delta)=d} 2a s\left(\frac{c}{\delta}, a; \frac{g}{\delta}, 0\right) \right) \right] \\
&= \sum_{\substack{\delta|N \\ (g=0)}} r_{\delta,0} \left[\frac{b\delta}{6} - \frac{c}{6\delta} + 2a s\left(\frac{c}{\delta}, a\right) \right] \\
&\hspace{10em} + \sum_{\substack{\delta|N \\ 0 < g < \delta}} \sum_{d|\delta} r_{\delta,d} \left[b\delta \left(\sum_{\left(\frac{g}{d}, \frac{\delta}{d}\right)=1} P_2\left(\frac{g/d}{\delta/d}\right) \right) - \phi\left(\frac{\delta}{d}\right) \frac{c}{6\delta} + \right. \\
&\hspace{15em} \left. \left(\sum_{\left(\frac{g}{d}, \frac{\delta}{d}\right)=1} 2a s\left(\frac{c}{\delta}, a; \frac{g/d}{\delta/d}, 0\right) \right) \right] \\
&= \sum_{\substack{\delta|N \\ (g=0)}} r_{\delta,0} \left[\frac{b\delta}{6} - \frac{c}{6\delta} + 2a s\left(\frac{c}{\delta}, a\right) \right] + \sum_{\substack{\delta|N \\ 0 < g < \delta}} \sum_{d|\delta} r_{\delta,d} \left[b\delta \left(\frac{d}{6\delta} \prod_{p|\frac{\delta}{d}} (1-p) \right) \right. \\
&\hspace{15em} \left. - \phi\left(\frac{\delta}{d}\right) \frac{c}{6\delta} + \left(\sum_{e|\frac{\delta}{d}} \mu(e) 2a s\left(\frac{c}{\delta} \frac{\delta}{de}, a\right) \right) \right].
\end{aligned}$$

Multiplying by $12c$ yields

$$\begin{aligned}
& \sum_{\substack{\delta|N \\ (g=0)}} r_{\delta,0} \left[2bc\delta - \frac{2c^2}{\delta} + 24ac s\left(\frac{c}{\delta}, a\right) \right] + \sum_{\substack{\delta|N \\ 0 < g < \delta}} \sum_{d|\delta} 2r_{\delta,d} bcd \prod_{p|\frac{\delta}{d}} (1-p) \\
& - \sum_{\substack{\delta|N \\ 0 < g < \delta}} \sum_{d|\delta} 2r_{\delta,d} \frac{c^2}{\delta} \phi\left(\frac{\delta}{d}\right) + \sum_{\substack{\delta|N \\ 0 < g < \delta}} \sum_{d|\delta} 24ac r_{\delta,d} \left(\sum_{e|\frac{\delta}{d}} \mu(e) s\left(\frac{c}{de}, a\right) \right) \\
& = \sum_{\substack{\delta|N \\ (g=0)}} 2r_{\delta,0} \left[bc\delta - \frac{c^2}{\delta} + \delta 12a \frac{c}{\delta} s\left(\frac{c}{\delta}, a\right) \right] \\
& + \sum_{\substack{\delta|N \\ 0 < g < \delta}} \sum_{d|\delta} 2r_{\delta,d} bcd \prod_{p|\frac{\delta}{d}} (1-p) \\
& - \sum_{\substack{\delta|N \\ 0 < g < \delta}} \sum_{d|\delta} 2r_{\delta,d} \frac{c^2}{\delta} \phi\left(\frac{\delta}{d}\right) \\
& + \sum_{\substack{\delta|N \\ 0 < g < \delta}} \sum_{d|\delta} 2r_{\delta,d} \left(\sum_{e|\frac{\delta}{d}} \mu(e) de 12a \frac{c}{de} s\left(\frac{c}{de}, a\right) \right).
\end{aligned}$$

Since $12hk s(h, k) \equiv (1 - k^2)(1 + h^2) \pmod{k}$ for $(h, k) = 1$ (see Lemma 10 of the Appendix of [20]), the last sum is, after reducing mod a ,

$$\begin{aligned}
& \sum_{\substack{\delta|N \\ (g=0)}} 2r_{\delta,0} \left[bc\delta - \frac{c^2}{\delta} + \delta(1 - a^2) \left(1 + \left(\frac{c}{\delta}\right)^2 \right) \right] \\
& + \sum_{\substack{\delta|N \\ 0 < g < \delta}} \sum_{d|\delta} 2r_{\delta,d} bcd \prod_{p|\frac{\delta}{d}} (1-p) - \sum_{\substack{\delta|N \\ 0 < g < \delta}} \sum_{d|\delta} 2r_{\delta,d} \frac{c^2}{\delta} \phi\left(\frac{\delta}{d}\right) \\
& + \sum_{\substack{\delta|N \\ 0 < g < \delta}} \sum_{d|\delta} 2r_{\delta,d} \left(\sum_{e|\frac{\delta}{d}} \mu(e) de (1 - a^2) \left(1 + \left(\frac{c}{de}\right)^2 \right) \right) \pmod{a} \\
& \equiv \sum_{\substack{\delta|N \\ (g=0)}} 2r_{\delta,0} (-1)\delta + \sum_{\substack{\delta|N \\ 0 < g < \delta}} \sum_{d|\delta} 2r_{\delta,d} (-1)d \prod_{p|\frac{\delta}{d}} (1-p) \\
& - \sum_{\substack{\delta|N \\ (g=0)}} 2r_{\delta,0} \frac{c^2}{\delta} - \sum_{\substack{\delta|N \\ 0 < g < \delta}} \sum_{d|\delta} 2r_{\delta,d} \frac{c^2}{\delta} \phi\left(\frac{\delta}{d}\right) + \sum_{\substack{\delta|N \\ (g=0)}} \sum_{d|\delta} 2r_{\delta,0} \delta \left(1 + \frac{c^2}{\delta^2} \right) \\
& + \sum_{\substack{\delta|N \\ 0 < g < \delta}} \sum_{d|\delta} 2r_{\delta,d} \left(\sum_{e|\frac{\delta}{d}} \mu(e) de \left(1 + \frac{c^2}{d^2 e^2} \right) \right) \pmod{a} \\
& \qquad \qquad \qquad (ad - bc = 1 \Rightarrow bc \equiv -1 \pmod{a})
\end{aligned}$$

$$\begin{aligned}
&= - \sum_{\substack{\delta|N \\ 0 < g < \delta}} \sum_{d|\delta} 2r_{\delta,d} d \prod_{p|\frac{\delta}{d}} (1-p) - \sum_{\substack{\delta|N \\ 0 < g < \delta}} \sum_{d|\delta} 2r_{\delta,d} \frac{c^2}{\delta} \phi\left(\frac{\delta}{d}\right) \\
&\quad + \sum_{\substack{\delta|N \\ 0 < g < \delta}} \sum_{d|\delta} 2r_{\delta,d} \left(\sum_{e|\frac{\delta}{d}} \mu(e) de \right) + \sum_{\substack{\delta|N \\ 0 < g < \delta}} \sum_{d|\delta} 2r_{\delta,d} \left(\sum_{e|\frac{\delta}{d}} \mu(e) \frac{c^2}{de} \right) \\
&= - \sum_{\substack{\delta|N \\ 0 < g < \delta}} \sum_{d|\delta} 2r_{\delta,d} d \prod_{p|\frac{\delta}{d}} (1-p) - \sum_{\substack{\delta|N \\ 0 < g < \delta}} \sum_{d|\delta} 2r_{\delta,d} \frac{c^2}{\delta} \phi\left(\frac{\delta}{d}\right) \\
&\quad + \sum_{\substack{\delta|N \\ 0 < g < \delta}} \sum_{d|\delta} 2r_{\delta,d} d \left(\sum_{e|\frac{\delta}{d}} \mu(e) e \right) + \sum_{\substack{\delta|N \\ 0 < g < \delta}} \sum_{d|\delta} 2r_{\delta,d} \frac{c^2}{d} \left(\sum_{e|\frac{\delta}{d}} \mu(e) e^{-1} \right) \\
&= - \sum_{\substack{\delta|N \\ 0 < g < \delta}} \sum_{d|\delta} 2r_{\delta,d} d \prod_{p|\frac{\delta}{d}} (1-p) - \sum_{\substack{\delta|N \\ 0 < g < \delta}} \sum_{d|\delta} 2r_{\delta,d} \frac{c^2}{\delta} \phi\left(\frac{\delta}{d}\right) \\
&\quad + \sum_{\substack{\delta|N \\ 0 < g < \delta}} \sum_{d|\delta} 2r_{\delta,d} d \prod_{p|\frac{\delta}{d}} (1-p) + \sum_{\substack{\delta|N \\ 0 < g < \delta}} \sum_{d|\delta} 2r_{\delta,d} \frac{c^2}{d} \prod_{p|\frac{\delta}{d}} (1-p^{-1})
\end{aligned}$$

since $\sum_{d|n} \mu(d) d^k = \prod_{p|n} (1-p^k)$. Finally, since $\phi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right)$ (Proposition 2.2.5 of [10]), we can write $\phi(\delta/d)$ as $\frac{\delta}{d} \prod_{p|\frac{\delta}{d}} (1-p^{-1})$, and hence, this last sum is

$$\begin{aligned}
&- \sum_{\substack{\delta|N \\ 0 < g < \delta}} \sum_{d|\delta} 2r_{\delta,d} \frac{c^2}{\delta} \phi\left(\frac{\delta}{d}\right) + \sum_{\substack{\delta|N \\ 0 < g < \delta}} \sum_{d|\delta} 2r_{\delta,d} \frac{c^2}{d} \prod_{p|\frac{\delta}{d}} (1-p^{-1}) \\
&\quad - \sum_{\substack{\delta|N \\ 0 < g < \delta}} \sum_{d|\delta} 2r_{\delta,d} \frac{c^2}{\delta} \phi\left(\frac{\delta}{d}\right) + \sum_{\substack{\delta|N \\ 0 < g < \delta}} \sum_{d|\delta} 2r_{\delta,d} \frac{c^2}{\delta} \left(\frac{\delta}{d} \prod_{p|\frac{\delta}{d}} (1-p^{-1}) \right) \\
&= 0.
\end{aligned}$$

Thus the multiplier is 1. □

To conclude if a product of generalized eta-functions is a modular form, we have to determine whether it is holomorphic at the cusps. An effective way to do this is given in [21]:

Theorem R. The order of the function $f(\tau)$ (defined in Theorem 1) at the cusp $\frac{\lambda}{\mu\epsilon}$ is

$$\frac{N}{2} \sum_{\substack{\delta|N \\ 0 < g < \delta}} r_{\delta,g} \frac{(\delta, \epsilon)^2}{\delta\epsilon} P_2\left(\frac{\lambda g}{(\delta, \epsilon)}\right).$$

In the next section, we will use Theorem 1 and Theorem R to create modular forms on $\Gamma_0(N)$. We end this section by showing how these Generalized Dedekind eta functions can be used to create a modular form of weight two similar to the Eisenstein series defined in chapter 3 of [13], by looking at their derivatives. Starting with the transformation formula

$$\eta_{\delta,g}\left(\frac{a\tau + b}{c\tau + d}\right) = \nu_{\delta,g}(A)\eta_{\delta,ag}(\tau)$$

where $g \neq 0$ (see chapter 3, section 2 of [13] to see what happens when $g = 0$), take the derivative of both sides with respect to τ :

$$\eta'_{\delta,g}\left(\frac{a\tau + b}{c\tau + d}\right)(c\tau + d)^{-2} = \nu_{\delta,g}(A)\eta'_{\delta,ag}(\tau),$$

or,

$$\eta'_{\delta,g}\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^2 \nu_{\delta,g}(A)\eta'_{\delta,ag}(\tau).$$

So, on $\Gamma_1(\delta)$, $\eta'_{\delta,g}(\tau)$ is a modular function of weight 2 with multiplier $\nu_{\delta,g}$. Since $\eta_{\delta,g}(\tau) \neq 0$ for any $\tau \in \mathfrak{h}$, we can divide $\eta'_{\delta,g}(\tau)$ by $\eta_{\delta,g}(\tau)$, to get a modular function of weight 2 on $\Gamma_1(\delta)$, without multiplier. In fact, examining $\eta'_{\delta,g}(\tau)/\eta_{\delta,g}(\tau)$ at the cusps, we find that this is a modular form:

Proposition 1. If f is a modular function of weight 0 with multiplier ν on Γ' , where Γ' is a congruence subgroup of Γ , then f' is a modular function of weight 2 with multiplier ν on Γ' . Furthermore, the order of f' at a cusp r is equal to the order of f at r .

Proof. If $\gamma \in \Gamma'$, then $f(\gamma z) = \nu(\gamma)f(z)$; taking the derivative of both sides, we get $f'(\gamma z)(cz + d)^{-2} = \nu(\gamma)f'(z)$, i.e.

$$f'(\gamma z) = \nu(\gamma)(cz + d)^2 f'(z).$$

For the behavior at the cusps, let $\gamma_0 \in \Gamma$ and suppose

$$f(\gamma_0 z) = \sum_{n \geq M} a_n q_N^n \quad (q_N = e^{2\pi iz/N}).$$

Then, taking the derivative of both sides, we get

$$f'(\gamma_0 z)(c_0 z + d_0)^{-2} = \sum_{n \geq M} a_n \frac{2\pi i n}{N} q_N^n,$$

$$\text{i.e. } f'(z)|[\gamma_0]_2 = \sum_{n \geq M} a_n \frac{2\pi i n}{N} q_N^n$$

where $f(z)|[\gamma]_k$ is the slash notation from [13]: $f(z)|[\gamma]_k = (\det \gamma)^{k/2} (cz+d)^{-k} f(\gamma z)$. Therefore f' is meromorphic at the cusp r (where $r = \gamma_0 \infty$), with the same order at r as f . \square

Corollary 1.1. If f is a modular function of weight 0 with multiplier on Γ' and $f \neq 0$ on h , then f'/f is a modular form of weight 2 on Γ' .

So, for $0 < g < \delta$, $\eta'_{\delta,g}(\tau)/\eta_{\delta,g}(\tau) \in M_2(\Gamma_1(\delta))$. Furthermore, if $\delta/g = 2, 3, 4$, or 6 , then $\eta'_{\delta,g}(\tau)/\eta_{\delta,g}(\tau) \in M_2(\Gamma_0(\delta))$.

We can find the q -expansion of $\eta'_{\delta,g}(\tau)/\eta_{\delta,g}(\tau)$ at infinity as follows: starting with

$$\eta_{\delta,g}(\tau) = q^{\frac{1}{2}\delta P_2(\frac{g}{\delta})} \prod_{\substack{m \equiv g \pmod{\delta} \\ m > 0}} (1 - q^m) \prod_{\substack{m \equiv -g \pmod{\delta} \\ m > 0}} (1 - q^m),$$

take the logarithmic derivative of both sides with respect to τ :

$$\begin{aligned} \frac{\eta'_{\delta,g}(\tau)}{\eta_{\delta,g}(\tau)} &= \frac{d}{d\tau} \log \eta_{\delta,g}(\tau) \\ &= \frac{d}{d\tau} \left(\frac{1}{2} \delta P_2 \left(\frac{g}{\delta} \right) (2\pi i \tau) + \sum_{\substack{m \equiv g \pmod{\delta} \\ m > 0}} \log(1 - q^m) + \sum_{\substack{m \equiv -g \pmod{\delta} \\ m > 0}} \log(1 - q^m) \right) \\ &= \frac{d}{d\tau} \left(\frac{1}{2} \delta P_2 \left(\frac{g}{\delta} \right) (2\pi i \tau) - \sum_{\substack{m \equiv g \pmod{\delta} \\ m > 0}} \sum_{n=1}^{\infty} \frac{q^{mn}}{n} - \sum_{\substack{m \equiv -g \pmod{\delta} \\ m > 0}} \sum_{n=1}^{\infty} \frac{q^{mn}}{n} \right) \\ &= \frac{1}{2} \delta P_2 \left(\frac{g}{\delta} \right) 2\pi i - \sum_{\substack{m \equiv g \pmod{\delta} \\ m > 0}} \sum_{n=1}^{\infty} 2\pi i m q^{mn} - \sum_{\substack{m \equiv -g \pmod{\delta} \\ m > 0}} \sum_{n=1}^{\infty} 2\pi i m q^{mn} \\ &= \pi i \delta P_2 \left(\frac{g}{\delta} \right) - 2\pi i \sum_{N=1}^{\infty} \sum_{\substack{m \equiv g \pmod{\delta} \\ N = mn}} m q^N - 2\pi i \sum_{N=1}^{\infty} \sum_{\substack{m \equiv -g \pmod{\delta} \\ N = mn}} m q^N \\ &= \pi i \delta P_2 \left(\frac{g}{\delta} \right) - 2\pi i \sum_{N=1}^{\infty} \left(\sum_{\substack{m \equiv g \pmod{\delta} \\ N = mn}} m + \sum_{\substack{m \equiv -g \pmod{\delta} \\ N = mn}} m \right) q^N \\ &= \pi i \delta P_2 \left(\frac{g}{\delta} \right) - 2\pi i \sum_{N=1}^{\infty} \sigma^{(\delta,g)}(N) q^N \end{aligned}$$

where $\sigma_k^{(\delta,g)}(N) = \sum_{\substack{d|N \\ d \equiv g \pmod{\delta}}} d^k + \sum_{\substack{d|N \\ d \equiv -g \pmod{\delta}}} d^k$ (and $\sigma_1^{(\delta,g)}(N)$ is simply denoted by $\sigma^{(\delta,g)}(N)$). The rearrangement of the above series is justified since $|q| < 1$ and hence the series $\sum_{\substack{m \equiv g \pmod{\delta} \\ m > 0}} \sum_{n=1}^{\infty} m q^{mn}$ and $\sum_{\substack{m \equiv -g \pmod{\delta} \\ m > 0}} \sum_{n=1}^{\infty} m q^{mn}$ converge absolutely:

$$\begin{aligned} \sum_{\substack{m \equiv g \pmod{\delta} \\ m > 0}} \sum_{n=1}^{\infty} m |q|^{mn} &= \sum_{\substack{m \equiv g \pmod{\delta} \\ m > 0}} m \sum_{n=1}^{\infty} |q|^{mn} \\ &= \sum_{\substack{m \equiv g \pmod{\delta} \\ m > 0}} m \frac{|q|^m}{1 - |q|^m}. \end{aligned}$$

Since $|q| < 1$, we can write $1 - |q| > \epsilon$ for some $\epsilon > 0$. We then also have $1 - |q|^m > 1 - |q| > \epsilon$, so

$$\sum_{\substack{m \equiv g \pmod{\delta} \\ m > 0}} m \frac{|q|^m}{1 - |q|^m} < \frac{1}{\epsilon} \sum_{\substack{m \equiv g \pmod{\delta} \\ m > 0}} m |q|^m,$$

which is finite by the ratio test:

$$\lim_{m \rightarrow \infty} \frac{(m+1)|q|^{m+1}}{m|q|^m} = |q| \lim_{m \rightarrow \infty} \frac{m+1}{m} = |q| < 1.$$

For simplicity, let $H_2^{(\delta,g)}(\tau)$ denote the normalization of the above series, i.e.

$$\begin{aligned} H_2^{(\delta,g)}(\tau) &= \frac{1}{\pi i \delta P_2(g/\delta)} \frac{\eta'_{\delta,g}(\tau)}{\eta_{\delta,g}} \\ &= 1 - \frac{2}{\delta P_2(g/\delta)} \sum_{N=1}^{\infty} \sigma^{(\delta,g)}(N) q^N. \end{aligned}$$

In conclusion, we have $H_2^{(\delta,g)}(\tau) \in M_2(\Gamma_1(\delta))$, and, if $\frac{\delta}{g} = 2, 3, 4$, or 6 , then $H_2^{(\delta,g)}(\tau) \in M_2(\Gamma_0(\delta))$.