I. Preliminary Results

In this section, we establish a few preliminary results which will be used later. The first three are corollaries to the following:

Lemma 1. Let $f : (0,1] \to \mathrm{C}$ and define

$$\mathrm{F}(m) = \sum_{\substack{1\ \ k\ \ m \\ (k,m)=1}} f\left(\frac{k}{m}\right),$$

and

$$F(n) = \sum_{k=1}^{n} f\left(\frac{k}{n}\right).$$

Then

$$\mathrm{F}(n) = \sum_{d|n} \mu(d) F\left(\frac{n}{d}\right).$$

Proof. This is a consequence of Möbius Inversion, since

$$\begin{aligned} F(n) &= \sum_{k=1}^{n} f\left(\frac{k}{n}\right) \\ &= \sum_{d|n} \sum_{\substack{1\ \ k\ \ n \\ (k,n)=d}} f\left(\frac{k}{n}\right) \\ &= \sum_{d|n} \sum_{\substack{1\ \ k\ \ n \\ (\frac{k}{d},\frac{n}{d})=1}} f\left(\frac{k/d}{n/d}\right) \\ &= \sum_{d|n} \mathrm{F}\left(\frac{n}{d}\right) \\ &= \sum_{d|n} \mathrm{F}(d). \end{aligned}$$

□

For the first corollary, we consider the second Bernoulli polynomial $P_2(x) := \{x\}^2 - \{x\} + \frac{1}{6}$ where $\{x\} = x - \lfloor x \rfloor$ denotes the fractional part of x.

Corollary 1. For $n \in \mathrm{N}$, we have

$$\sum_{\substack{1 \leq k \leq n \\ (k,n)=1}} P_2\left(\frac{k}{n}\right) = \frac{1}{6n} \prod_{\substack{p|n \\ p \text{ prime}}} \left(1-p\right).$$

More generally, for d a divisor of n,

$$\sum_{\substack{1 \leq k \leq n \\ (k,n)=d}} P_2\left(\frac{k}{n}\right) = \frac{d}{6n} \prod_{\substack{p|\frac{n}{d} \\ p \text{ prime}}} \left(1-p\right).$$

Proof. One can easily verify (from the definition) that $\sum_{k=1}^{n} P_2\left(\frac{k}{n}\right) = \frac{1}{6n}$ and $\sum_{d|n} \mu(d)d^r = \prod_{p|n}(1-p^r)$. The result now follows from Lemma 1, with $f(x) = P_2(x)$.

☐

For the second corollary, we consider a special case of the Meyer sum. The Meyer sum is defined as

$$s(h,k;x,y) = \sum_{\mu (\text{mod } k)} \left(\left(\frac{\mu + y}{k}\right)\right) \left(\left(h\frac{\mu + y}{k} + x\right)\right)$$

where $h, k \in \mathrm{Z}$, $x, y \in \mathrm{R}$, and

$$((z)) = \begin{cases} \{z\} - \frac{1}{2} & \text{if } z \notin \mathrm{Z} \\ 0 & \text{if } z \in \mathrm{Z} \end{cases}$$

(here, $\{z\} = z - \lfloor z \rfloor$ denotes the fractional part of z). We will be interested in Meyer sums of the form $s(c, a; \frac{g}{\delta}, 0)$.

If $(a, nc) = 1$, then one can show (from the definition) that

$$\sum_{k=1}^{n} s\left(a, c\,; 0, \frac{k}{n}\right) = s(a, nc) \tag{1}$$

where $s(h,k)$ denotes the usual Dedekind sum (so $s(h,k) = s(h,k;0,0)$). The reciprocity law for Meyer sums states that, if $(h,k) = 1$, then

$$\begin{aligned} &s(h,k;x,y) + s(k,h;y,x) \\ &\qquad = -\frac{1}{4}\delta(x)\delta(y) + ((x))((y)) + \frac{1}{2}\left[\frac{h}{k}P_2(y) + \frac{1}{hk}P_2(hy+kx) + \frac{k}{h}P_2(x)\right] \end{aligned}$$

where

$$\delta(z) = \begin{cases} 1 & \text{if } z \in \mathrm{Z} \\ 0 & \text{if } z \notin \mathrm{Z} \end{cases}$$

([20], ch. 5). Since $(a, c) = 1$, we have (using $h = a, k = c, x = 0, y = \frac{k}{n}$)

$$s\Big(a, c\,; 0, \frac{k}{n}\Big) = -s\Big(c, a\,; \frac{k}{n}, 0\Big) - \frac{1}{4}\delta\Big(\frac{k}{n}\Big) + \frac{a}{2c}P_2\Big(\frac{k}{n}\Big) + \frac{1}{2ac}P_2\Big(\frac{ak}{n}\Big) + \frac{c}{12a}; \tag{2}$$

and since $(a, n) = 1$ we also have (using $h = a, k = c, x = 0 = y$)

$$s(a, nc) = -s(nc, a) - \frac{1}{4} + \frac{a}{12nc} + \frac{1}{12nac} + \frac{nc}{12a}. \tag{3}$$

So, combining (1), (2), and (3)

$$\begin{aligned} -s(nc, a) - \frac{1}{4} + \frac{a}{12nc} + \frac{1}{12nac} + \frac{nc}{12a} \\ &= \sum_{k=1}^{n}\left[-s\Big(c, a\,; \frac{k}{n}, 0\Big) - \frac{1}{4}\delta\Big(\frac{k}{n}\Big) + \frac{a}{2c}P_2\Big(\frac{k}{n}\Big) + \frac{1}{2ac}P_2\Big(\frac{ak}{n}\Big) + \frac{c}{12a}\right] \\ &= \left[\sum_{k=1}^{n} -s\Big(c, a\,; \frac{k}{n}, 0\Big)\right] - \frac{1}{4} + \frac{a}{2c}\frac{1}{6n} + \frac{nc}{12a} + \frac{1}{2ac}\sum_{k=1}^{n} P_2\Big(\frac{ka}{n}\Big); \end{aligned}$$

since $(a, n) = 1$, we have $\displaystyle\sum_{k=1}^{n} P_2\Big(\frac{ka}{n}\Big) = \sum_{k=1}^{n} P_2\Big(\frac{k}{n}\Big) = \frac{1}{6n}$, so the above becomes

$$\sum_{k=1}^{n} s\Big(c, a\,; \frac{k}{n}, 0\Big) = s(nc, a).$$

Now, letting $\mathrm{F}(m) = \displaystyle\sum_{\substack{1 \le k \le m \\ (k,m)=1}} s\Big(c, a; \frac{k}{m}, 0\Big)$ and $F(n) = \displaystyle\sum_{k=1}^{n} s\Big(c, a; \frac{k}{n}, 0\Big) = s(nc, a)$, we get

Corollary 2. If $(a, nc) = 1$, then

$$\sum_{\substack{1 \le k \le n \\ (k,n)=1}} s\Big(c, a\,; \frac{k}{n}, 0\Big) = \sum_{d|n} \mu(d) s\Big(\frac{nc}{d}, a\Big).$$

More generally, for d a divisor of n, we have

$$\sum_{\substack{1 \le k \le n \\ (k,n)=d}} s\Big(c, a\,; \frac{k}{n}, 0\Big) = \sum_{e|\frac{n}{d}} \mu(e) s\Big(\frac{nc}{de}, a\Big).$$

For the third corollary, we let $f(x) = \lfloor ax \rfloor$, where $a \in$ Z. In particular, let $(a, n) = 1$ and consider

$$\mathrm{F}(m) = \sum_{\substack{1 \le k \le m \\ (k,m)=1}} \left\lfloor a\frac{k}{m} \right\rfloor;$$

since

$$\begin{aligned} F(n) &= \sum_{k=1}^{n} \left\lfloor a\frac{k}{n} \right\rfloor \\ &= \frac{1}{2}\left(\sum_{k=1}^{n} \left\lfloor a\frac{k}{n} \right\rfloor + \sum_{k=0}^{n-1} \left\lfloor a\frac{n-k}{n} \right\rfloor \right) \\ &= \frac{1}{2}\left(2a + \sum_{k=1}^{n-1} \left(\left\lfloor a\frac{k}{n} \right\rfloor + \left\lfloor a - a\frac{k}{n} \right\rfloor \right) \right) \\ &= \frac{1}{2}\left(2a + \sum_{k=1}^{n-1} \left(\left\lfloor a\frac{k}{n} \right\rfloor + a + \left\lfloor -a\frac{k}{n} \right\rfloor \right) \right) \\ &= \frac{1}{2}\left(2a + \sum_{k=1}^{n-1} (a-1) \right) \qquad (\lfloor x \rfloor + \lfloor -x \rfloor = -1 \text{ for } x \notin \mathrm{Z}) \\ &= \frac{a-1}{2}(n+1) + 1, \end{aligned}$$

we get

Corollary 3. If $(a, n) = 1$ and $n > 1$, then

$$\sum_{\substack{1 \le k \le n \\ (k,n)=1}} \left\lfloor a\frac{k}{n} \right\rfloor = \frac{a-1}{2}\phi(n).$$

Proof. By Lemma 1, we have

$$\begin{aligned} \sum_{\substack{1 \le k \le n \\ (k,n)=1}} \left\lfloor a\frac{k}{n} \right\rfloor &= \sum_{d \mid n} \mu(d) F\left(\frac{n}{d}\right) \\ &= \sum_{d \mid n} \mu(d) \left[\frac{a-1}{2}\left(\frac{n}{d} + 1\right) + 1 \right] \\ &= \frac{a-1}{2} \sum_{d \mid n} \mu(d) \frac{n}{d} \\ &= \frac{a-1}{2}\phi(n). \end{aligned}$$

□

Note: One can also prove Corollary 3 directly by using the fact that $\lfloor x \rfloor + \lfloor -x \rfloor = -1$ for $x \notin \mathbb{Z}$.

For the final result, recall that the Legendre symbol $\left(\frac{b}{a}\right)$ is defined as follows: if p is an odd prime, then $\left(\frac{b}{p}\right)$ is defined as

$$\left(\frac{b}{p}\right) = \begin{cases} 0 & \text{if } b \equiv 0 \pmod{p} \\ 1 & \text{if } b \text{ is a (nonzero) square modulo } p \\ -1 & \text{if } b \text{ is not a square modulo } p. \end{cases}$$

The definition is then extended to $\left(\frac{b}{a}\right)$, where a is odd, by taking the prime factorization of a: $a = p_1^{\alpha_1} \cdots p_s^{\alpha_s}$, and defining $\left(\frac{b}{a}\right) = \left(\frac{b}{p_1}\right)^{\alpha_1} \cdots \left(\frac{b}{p_s}\right)^{\alpha_s}$ (see [10]).

Lemma 2. Let $D \in \mathbb{N}$ with prime factorization $D = p_1^{\alpha_1} \cdots p_t^{\alpha_t}$, and let $a \in \mathbb{N}$ be relatively prime to D. Then $\frac{1}{2} \sum_{D=ef} \mu(e)\left(\frac{f}{a}\right) \equiv 1 \pmod{2}$ if and only if $t = 1$ and $\left(\frac{p}{a}\right) = -1$, i.e. $D = p^\alpha$ for some prime p with $\left(\frac{p}{a}\right) = -1$.

Proof. Since the function μ and the Legendre function $\left(\frac{\cdot}{a}\right)$ are multiplicative, so is $F(n) = \sum_{n=ef} \mu(e)\left(\frac{f}{a}\right)$. Thus, F is determined by its values on powers of primes. For p^α, we get

$$\begin{aligned} F(p^\alpha) &= \mu(1)\left(\frac{p^\alpha}{a}\right) + \mu(p)\left(\frac{p^{\alpha-1}}{a}\right) + \cdots + \mu(p^\alpha)\left(\frac{1}{a}\right) \\ &= \left(\frac{p^\alpha}{a}\right) - \left(\frac{p^{\alpha-1}}{a}\right) \\ &= \left(\frac{p^\alpha}{a}\right)\left[1 - \left(\frac{p}{a}\right)\right]. \end{aligned}$$

Thus,

$$\begin{aligned} \frac{1}{2} \sum_{D=ef} \mu(e)\left(\frac{f}{a}\right) &= \frac{1}{2} \prod_{\substack{p \mid D \\ p \text{ prime}}} \left(\frac{p}{a}\right)^\alpha \left[1 - \left(\frac{p}{a}\right)\right] \\ &= \frac{1}{2}\left(\frac{D}{a}\right) \prod_{\substack{p \mid D \\ p \text{ prime}}} \left[1 - \left(\frac{p}{a}\right)\right]. \end{aligned}$$

If $\left(\frac{p}{a}\right) = 1$ for any prime p dividing D, then $\frac{1}{2} \sum_{D=ef} \mu(e)\left(\frac{f}{a}\right) = 0$. Suppose that

$\left(\frac{p}{a}\right) = -1$ for all primes p dividing D; then

$$\begin{aligned}\frac{1}{2}\left(\frac{D}{a}\right) \prod_{\substack{p\mid D\\ p\text{ prime}}} \left[1 - \left(\frac{p}{a}\right)\right] &= \frac{1}{2}\left(\frac{D}{a}\right)2^t\\ &= 2^{t-1}\left(\frac{D}{a}\right).\end{aligned}$$

The only way that this integer could be odd is if $t = 1$, as claimed. $\square$