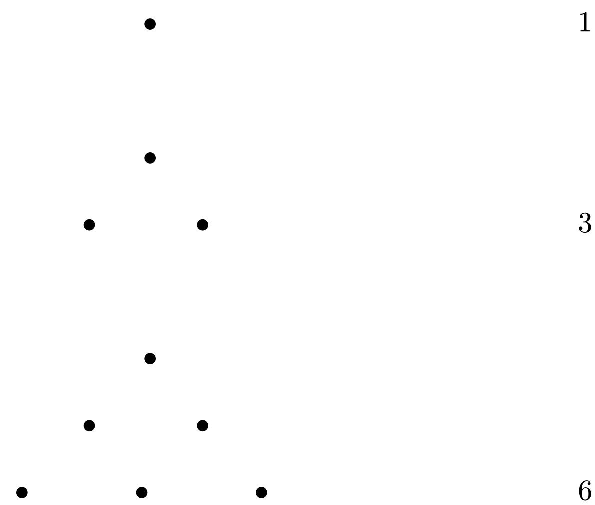
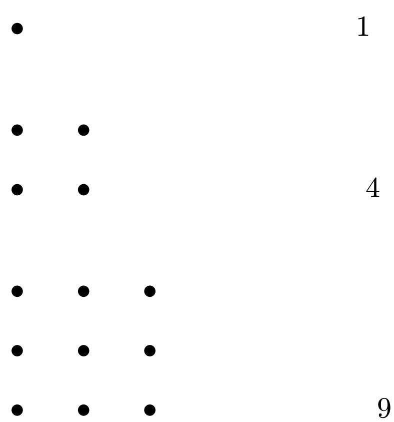
Introduction

The study of figurate numbers has a long and varied history (see, for example, Chapter 1 of [5]); similarly, the study of modular/elliptic functions has produced a great deal of literature. One example where the two subjects overlap is the study of the classical theta function (see the beginning of Section III). In this thesis, we will use modular forms to examine the representation of an integer as a sum of pentagonal and octagonal numbers.

A triangular number is a number which can be represented as a triangular array of nodes in the way depicted below:



and so forth. A square number (or square) is a number which can be represented as a square array of nodes in the way depicted below:



and so forth. This can be extended to figures with any number of sides.

In general, the formula for the nth $m-$gonal number is $\frac{m-2}{2}n^2 - \frac{m-4}{2}n$. If n is positive, then we say that $\frac{m-2}{2}n^2 - \frac{m-4}{2}n$ is an $m-$gonal number of positive rank; and if n is negative, then $\frac{m-2}{2}n^2 - \frac{m-4}{2}n$ is an $m-$gonal number of negative rank. (We can similarly define $m-$gonal numbers of non-negative or non-positive rank.) For triangular numbers and squares, the negative rank numbers do not contribute anything new; for example, the (-3)rd triangular number is the same as the 2nd triangular number $(\frac{1}{2}(-3)^2 + \frac{1}{2}(-3) = 3 = \frac{1}{2}(2)^2 + \frac{1}{2}(2))$, and the (-3)rd square is the same as the 3rd square $((-3)^3 = (3)^2)$. However, for $m \geq 5$, the negative rank $m-$gonal numbers are distinct from the positive rank numbers. For example, the positive rank pentagaonal numbers are

$$1, \quad 5, \quad 12, \quad 22, \quad 35, \quad 51, \quad \ldots$$

while the negative rank pentagonal numbers are

$$2, \quad 7, \quad 15, \quad 26, \quad 40, \quad 57, \quad \ldots,$$

and the positive rank octagonal numbers are

$$1, \quad 8, \quad 21, \quad 40, \quad 65, \quad 96, \quad \ldots$$

while the negative rank octagonal numbers are

$$5, \quad 16, \quad 33, \quad 56, \quad 85, \quad 120, \quad \ldots.$$

One can show that the positive rank hexagonal numbers correspond to the triangular numbers with odd (positive) rank, while the negative rank hexagonal numbers correspond to the triangular numbers with even (positive) rank. In this paper, we will look at sums of general rank pentagonal and octagonal numbers.

Fermat believed that any positive integer N can be written as a sum of m $m-$gonal numbers ([5], p. 6). This would mean, for example, any positive integer could be written as a sum of three triangular numbers (hence, Gauss' journal entry, "ΕΥΡΗΚΑ! num $= \Delta + \Delta + \Delta$" – [3]) or four squares (proven by Legendre – [15], p. 5). Cauchy eventually proved the general statement, in fact proving even more:

> Let $m \geq 5$ and $N \geq 28(m-2)^3$. If m is odd, then N is the sum of four $m-$gonal numbers. If m is even, then N is the sum of five $m-$gonal numbers, at least one of which is 0 or 1. ([15], Theorem 1.10)

Thus, every integer greater than $28(5-2)^3 = 756$ can be written as a sum of four pentagonal numbers of positive rank. Clearly then, every integer greater than 756 can be written as a sum of four general rank pentagonal numbers. (In fact, one can show that every positive integer can be written as a sum of three general rank pentagonal numbers – see [8], eq. (4.10).) We will show that

$$\frac{1}{2}\sigma(6N+1) \square p_4(N) \square \sigma(6N+1),$$

where $p_4(N)$ denotes the number of ways of writing N as a sum of four general rank pentagonal numbers and σ is the divisor sum function: $\sigma(n) = \sum_{d|n} d$.

Also, Cauchy's result says that every integer greater than $28(8-2)^3 = 6048$ can be written as a sum of five positive rank octagonal numbers, at least one of which is 0 or 1. Thus, every integer greater than 6048 can be written as a sum of at most five general rank octagonal numbers, at least one of which is 0 or 1. We will show that, unlike in the case of pentagonal numbers, there is no increasing lower bound for $o_4(N)$, where $o_4(N)$ denotes the number of ways of writing N as a sum of four general rank octagonal numbers. In fact, $o_4(N) = 1$ for infinitely many values of N.

To prove these assertions, we will use modular forms; specifically, we will use the generalized Dedekind eta function (defined in Section II) and something of a generalization of a result of S. Robins [21] to construct these forms. We will mainly be interested in modular forms on $\Gamma_0(N)$ (also defined in Section 11), usually of weight k, where k is even, although we will give a brief discussion of modular forms of odd weight in Section IV.

We will conclude with a summary of other possible applications. These include representations with other figurate numbers, hypergeometric series, and some specialized partitions.