Curious Consequences of a Miscopied Quadratic

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Motivation and Preliminaries

A problem on a student's algebra exam read, "Factor the polynomial x^2-5x+6." One student incorrectly copied the quadratic as x^2-5x-6, and then proceeded to factor it as $(x-6)(x+1)$. Since this altered polynomial factored nicely, the student had no reason to suspect that something was amiss. (Students generally assume that instructors give nice problems.) Of course, the original also factored nicely.

This episode raises the question of when two quadratic expressions, x^2-bx+c and x^2-bx-c, both factor over the integers $\mathbb{Z}$. (Note that changing the sign of b, rather than c, simply changes the signs of the roots of the original polynomial.) When both polynomials $x^2-bx\pm c$ have integer roots, we call them a *compatible* pair. We investigate these objects in this paper.

A search for compatible pairs sorted by the size of b yields the following:

$$x^2-5x\pm 6,\quad x^2-10x\pm 24,\quad x^2-13x\pm 30,\quad x^2-15x\pm 54,\quad x^2-17x\pm 60,$$

$$x^2-20x\pm 96,\quad x^2-25x\pm 84,\quad x^2-25x\pm 150,\quad x^2-26x\pm 120,.\ \ldots\ .$$

We observe that if a quadratic x^2-bx+c (without restriction on the sign of c) is factorable over $\mathbb{Z}$, then so is $x^2-bdx+cd^2$ for any integer d. This fact allows us to organize our previous examples into families of equivalent compatible pairs:

$$\begin{array}{llllll}
x^2-5x\pm 6, & x^2-10x\pm 24, & x^2-15x\pm 54, & x^2-20x\pm 96, & x^2-25x\pm 150, & \cdots \\
x^2-13x\pm 30, & x^2-26x\pm 120, & \cdots & & & \\
x^2-17x\pm 60, & \cdots & & & &
\end{array}$$

$x^2 - 25x \pm 84,\ .\quad \cdots$

Based on our observation, we can consider the first entry in each row as generating the entire family in that row. Note that, for each of these generators, b and c are relatively prime. We define a compatible pair with this property to be *primitive*.

To continue our search for compatible pairs, we now employ the quadratic formula. It tells us that the solutions of the equation $x^2 - bx - c = 0$ are $x = \dfrac{b \pm \sqrt{b^2 + 4c}}{2}$. For the roots of $x^2 - bx - c$ to be integers, $b^2 + 4c$ must be a perfect square, say k^2. Similarly, from $x^2 - bx + c = 0$, we find that $b^2 - 4c$ must be a perfect square, say l^2. This gives rise to the equations

$$k^2 = b^2 + 4c \text{ and } l^2 = b^2 - 4c,$$

which, in turn, yield the following necessary conditions on b and c:

$$2b^2 = k^2 + l^2 \tag{1}$$

$$8c = k^2 - l^2. \tag{2}$$

It turns out that (2) does not help us, so we focus on (1). We now look for two distinct squares whose sum is twice a square. For example, $7^2 + 1^2 = 2 \cdot 5^2$, so we get a compatible pair with $b = 5$ and $c = \frac{1}{8}\left(7^2 - 1^2\right) = 6$. This is the first compatible pair, $x^2 - 5x \pm 6$, from the exam.

Pythagorean Triples

Our search for compatible pairs can be viewed as a search for Pythagorean triples: positive integer solutions to the equation $X^2 + Y^2 = Z^2$. Specifically, there is a one-to-one correspondence between the values of k and l that lead to compatible pairs and Pythagorean triples:

> If k and l yield a compatible pair $x^2 - bx \pm c$, then $\left(\frac{k-l}{2}, \frac{k+l}{2}, b\right)$ is a Pythagorean triple.
>
> Conversely, if (X, Y, Z) is a Pythagorean triple, then $k = X + Y$ and $l = X - Y$ lead to a compatible pair.

This can be derived from the correspondence between solutions (k, l) to equation (1) and Pythagorean triples mentioned in [**2**, p. 439]. This one-to-one correspondence, in fact, gives a one-to-one correspondence between primitive compatible pairs and primitive Pythagorean triples (triples in which X, Y, and Z are pairwise relatively prime).

Theorem 1. *A compatible pair is primitive if and only if the corresponding Pythagorean triple is primitive.*

Proof. Let $x^2 - bx \pm c$ be a primitive compatible pair, arising from the integers k and l. Suppose (by way of contradiction) that the Pythagorean triple $\left(\frac{k-l}{2}, \frac{k+l}{2}, b\right)$ is not primitive. In particular, let p be a prime divisor of $\frac{k-l}{2}, \frac{k+l}{2}$, and b. Then p^2 will divide $\frac{k-l}{2} \cdot \frac{k+l}{2}$. It then follows from (2) that c must be divisible by p, which contradicts our assumption that $x^2 - bx \pm c$ is primitive.

For the converse, let $\left(\frac{k-l}{2}, \frac{k+l}{2}, b\right)$ be a primitive Pythagorean triple. Since the hypotenuse of a primitive Pythagorean triple must be odd (consider the equation $X^2 + Y^2 = Z^2$

modulo 4). Now suppose (by way of contradiction) that p is an odd prime that divides both b and c. Then, by (1) and (2), p divides both $k^2 + l^2$ and $k^2 - l^2$. Thus, p divides both their sum and difference, so it divides both $2k^2$ and $2l^2$. However, since p is an odd prime, it must therefore be a divisor of both k and l. Hence p divides both $\frac{k-l}{2}$ and $\frac{k+l}{2}$, contradicting the fact that the triple $\left(\frac{k-l}{2}, \frac{k+l}{2}, b\right)$ is primitive. □

We see now that the problem of finding (primitive) compatible pairs is equivalent to that of finding (primitive) Pythagorean triples. Fortunately, the subject of Pythagorean triples has been well-studied. For example, from Theorem 3.20 of [**2**], we have the following result.

Pythagorean Triple Fact 1: *A positive integer is the hypotenuse of a Pythagorean triple if and only if it has at least one prime factor congruent to 1 modulo 4.*

This is a consequence of the Pythagorean Triple Fact 3 below, so we skip the proof. For $x^2 - bx \pm c$ to be a compatible pair, there must be a corresponding Pythagorean triple with b as the hypotenuse. This gives the following corresponding fact.

Compatible Pair Fact 1: *If b has a prime factor congruent to 1 modulo 4, then there is a compatible pair $x^2 - bx \pm c$.*

Here are the eight smallest positive integers b with such a prime factor and the corresponding value of c.

b	5	10	13	15	17	20	25	26
c	6	24	30	54	60	96	84	120

Notice that not all of these compatible pairs are primitive. When will they be primitive?

Pythagorean Triple Fact 2: *A positive integer is the hypotenuse of a primitive Pythagorean triple if and only if its prime factorization includes only primes congruent to 1 modulo 4.*

This follows from Pythagorean Triple Fact 4 below. As above, for $x^2 - bx \pm c$ to be a compatible pair, there must be a corresponding Pythagorean triple with *b* as the hypotenuse.

Compatible Pair Fact 2: *If the prime factorization of b includes only primes congruent to 1 modulo 4, then there is a primitive compatible pair* $x^2 - bx \pm c$.

The table below lists the six smallest positive integers having every prime factor congruent to 1 modulo 4, along with the corresponding value of *c*.

b	5	13	17	25	29	37
c	6	30	60	85	210	210

An interesting thing happens for $b = 65$; there are four quadratic pairs:

$$x^2 - 65x \pm 1014, \quad x^2 - 65x \pm 750, \quad x^2 - 65x \pm 504, \quad \text{and} \quad x^2 - 65x \pm 924.$$

Of these, only the last two are primitive. We now consider how many compatible pairs there are for a given value of *b*, and of those, how many are primitive.

Pythagorean Triple Fact 3: *Let the prime factorization of n be* $2^{\gamma}\prod_{i=1}^{s} p_i^{\alpha_i} \prod_{j=1}^{t} q_j^{\beta_j}$ *as above. Then the number of Pythagorean triples with hypotenuse n is* $\frac{1}{2}\left[\prod_{i=1}^{s}(2\alpha_i + 1) - 1\right]$.

This formula is given in [**1**] as Formula 3 on p. 117. Beiler does not prove this formula; however a proof can be deduced from Theorem 3.22 of [**2**]. Niven, Zuckerman, and Montgomery use their function $R(n)$ to count the number of integer solutions to the equation $x^2 + y^2 = n$; thus, $R(n^2)$ will count the number of solutions to $x^2 + y^2 = n^2$. We want to use this to count the number of Pythagorean triples with hypotenuse *n*; however, it counts more than we want. Beginning with the formula $R\left(n^2\right)=4\prod_{i=1}^{s}\left(2\alpha_i+1\right),$ we

- first eliminate the four solutions $\left(0\right)^2+\left(\pm n\right)^2=n^2$ and $\left(\pm n\right)^2+\left(0\right)^2=n^2,$ as these do not lead to Pythagorean triples;
- next divide by 4 to adjust for the four possible variations in the signs $\left(\pm x\right)^2+\left(\pm y\right)^2=n^2,$ since Pythagorean triples consists of positive integer side lengths;
- finally, divide by 2 to adjust for the different orderings of *x* and *y*.

This gives us Beiler's formula. Note that there will be a Pythagorean triple with hypotenuse *n* if and only if $s\geq 1,$ which proves Pythagorean Triple Fact 1.

From this fact about Pythagorean triples, we have the corresponding fact about compatible pairs.

Compatible Pair Fact 3: *If b has the prime factorization* $2^{\gamma}\prod_{i=1}^{s}p_i^{\alpha_i}\prod_{j=1}^{t}q_j^{\beta_j}$ *as above, then the number of compatible pairs* $x^2-bx\pm c$ *is* $\frac{1}{2}\left[\prod_{i=1}^{s}\left(2\alpha_i+1\right)-1\right].$

For example, for $b=65,$ we have $s=2$ and $\alpha_1=\alpha_2=1,$ so there should be four compatible pairs—which we found earlier. In fact, whenever *b* is the product of two distinct primes, both congruent to 1 modulo 4, there will always be four compatible pairs.

Finally, we look at primitive compatible pairs.

Pythagorean Triple Fact 4: *Let the prime factorization of n be* $2^{\gamma}\prod_{i=1}^{s} p_i^{\alpha_i}\prod_{j=1}^{t} q_j^{\beta_j}$ *as above. Then the number of primitive Pythagorean triples with hypotenuse n is 0 if* $\alpha > 0$ *or* $t > 0$, *and* 2^{s-1} *otherwise.*

This fact is given (without proof) on p. 117 of [**1**]. A proof can be constructed from Theorem 3.22 of [**2**] using their function *r* (dividing by 8 to adjust for signs and orderings).

Compatible Pair Fact 4: *Suppose that b has the prime factorization* $2^{\gamma}\prod_{i=1}^{s} p_i^{\alpha_i}\prod_{j=1}^{t} q_j^{\beta_j}$ *as above. If* $\alpha > 0$ *or* $t > 0$, *then there are no primitive compatible pairs for b; otherwise, we have* $b = \prod_{i=1}^{s} p_i^{\alpha_i}$, *and there are* 2^{s-1} *primitive compatible pairs.*

For example, if $b = 65$, then $s = 2$, there are two primitive compatible pairs as we found earlier.

Exercises for the Reader

If there are 13 compatible pairs corresponding to the integer *b*, how many of them can be primitive? What is the smallest positive integer *b* which has 13 compatible pairs? If you have a calculator (or better yet, a computer), find all of the compatible pairs for this value of *b*. Then find all of the Pythagorean triples with this value of *b* as the hypotenuse. (The solutions can be found at http://staff.mwsc.edu/~vestal/talks.html.)

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Bibliography

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2. I. Niven, H. S. Zuckerman, and H. L. Montgomery, *An Introduction to the Theory of Numbers*, 5th ed., John Wiley & Sons, 1991.