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Entropy and volume growth

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Abstract. We consider a C^{1+1} diffeomorphism f of a compact manifold M which preserves an ergodic probability measure μ. We conclude that μ-a.e. $x \in M$ is contained in a disk $D_x \subset W^u(x)$, with D_x open in the $W^u(x)$ topology, which exhibits an exponential volume growth rate greater than or equal to the measure-theoretic entropy of f with respect to μ. Drawing on results of Newhouse and Yomdin, we then find that when f is C^∞ and μ is a measure of maximal entropy, this exponential volume growth rate equals the topological entropy of f for μ-a.e. x.

1. *Introduction*

Let f be a diffeomorphism of a compact manifold M which preserves an ergodic probability measure μ. We say f is $C^{1+\alpha}$ if f and f^{-1} have α-Hölder derivatives, and f is C^{1+1} if f and f^{-1} have Lipschitz derivatives. The measure-theoretic entropy of f with respect to μ will be denoted by h_μ, while h will denote the topological entropy of f. The global unstable manifold of a point x will be denoted by $W^u_{\text{glob}}(x)$. Letting $V(A)$ denote the volume of a submanifold A computed with respect to the induced metric on A, we define the exponential volume growth rate of A as

$$G(A) \equiv \liminf_{n\to\infty} \frac{1}{n} \log \frac{V(f^n A)}{V(A)}.$$

Also define

$$\Gamma(A) \equiv \lim_{n\to\infty} \frac{1}{n} \log \frac{V(f^n A)}{V(A)}$$

whenever this limit exists.

In [**4**], Newhouse establishes that

$$\sup_D \limsup_{n\to\infty} \frac{1}{n} \log \frac{V(f^n(D))}{V(D)} \geq h$$

for $C^{1+\alpha}$ self-maps of M, where the supremum is taken over disks D transverse to stable manifolds. He extends these results in [**5**], and incorporates ideas developed by Yomdin in [**10**] in the C^∞ case to prove:

(i) there exists a C^∞ disk D such that

$$\lim_{n\to\infty} \frac{1}{n} \log \frac{V(f^n(D))}{V(D)} = h,$$

(ii) the map $\mu \to h_\mu(f)$ is uppersemicontinuous, and
(iii) the map $f \to h(f)$ is uppersemicontinuous.

In this paper we develop analogues of Newhouse's volume growth rate results for C^{1+1} diffeomorphisms, the essential differences being that we need only consider disks contained in unstable manifolds, and our results take the form of a.e.-existence as opposed to simple existence. Specifically, we will prove the following theorem.

THEOREM 1.1. *Let f be a C^{1+1} diffeomorphism of a compact manifold M which preserves an ergodic probability measure μ. Then for μ-a.e. $x \in M$, there exists a disk D_x, open in the submanifold topology of $W^u_{\text{glob}}(x)$, with $x \in D_x \subset W^u_{\text{glob}}(x)$ and $G(D_x) \geq h_\mu$.*

COROLLARY 1.1. *If f is C^{1+1}, then there exist $A \subset M$ of measure one with respect to any f-invariant probability measure and a disk D_x, open in the submanifold topology of $W^u_{\text{glob}}(x)$, with $x \in D_x \subset W^u_{\text{glob}}(x)$ for each $x \in A$ such that* $\sup_{x\in A} G(D_x) \geq h$.

COROLLARY 1.2. *If f is C^∞, there exists a disk D such that $\Gamma(D) = h$. Further, if μ is a measure of maximal entropy, then for μ-a.e. $x \in M$, there exists a disk D_x, open in the submanifold topology of $W^u_{\text{glob}}(x)$, with $x \in D_x \subset W^u_{\text{glob}}(x)$ and $\Gamma(D_x) = h$.*

2. *Preliminaries*

Consider a diffeomorphism f acting on a manifold M, together with an f-invariant, ergodic probability measure μ defined on the Borel sets of M.

Given a metric ρ defined on a neighborhood of x in $W^u_{\text{glob}}(x)$, the local unstable manifold of radius r with respect to the metric ρ of a point x will be denoted by $W^u_{\text{loc}}(x, r, \rho)$. $W^u_{\text{loc}}(x)$ will be used on those occasions when it is unnecessary to specify a radius and metric. We also define $B^u(x, r, \rho) \subset W^u_{\text{glob}}(x)$ as being the ball, open in the unstable manifold topology, about x of radius r with respect to a metric ρ.

We may use the multiplicative ergodic theorem of Oseledec with respect to our invariant measure μ to conclude the following:

MET-1 for μ-a.e. $x \in M$, there exist f-invariant non-negative integer-valued functions σ and υ; f-invariant positive integer-valued functions

$$d_{-\sigma}, \dots, d_{-1}, d_0, d_1, \dots, d_\upsilon;$$

and f-invariant real-valued functions

$$-\infty < \lambda_{-\sigma} < \cdots < \lambda_{-1} < \lambda_0 = 0 < \lambda_1 < \cdots < \lambda_\upsilon < \infty;$$

MET-2 for μ-a.e. $x \in M$, there exists an f-invariant splitting of $T_x(M)$ as

$$\mathbf{E}^x_{-\sigma} \oplus \cdots \oplus \mathbf{E}^x_{-1} \oplus \mathbf{E}^x_0 \oplus \mathbf{E}^x_1 \oplus \cdots \mathbf{E}^x_\upsilon;$$

MET-3 for μ-a.e. $x \in M$, $\dim(\mathbf{E}^x_i) = d_i$ for $-\sigma \leq i \leq \upsilon$;

MET-4 for μ-a.e. $x \in M$, $-\sigma \le i \le \upsilon$, and any non-zero $v \in \mathbf{E}^x$,

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|D_x f^n(v)\| = \lambda_i;$$

MET-5 for μ-a.e. $x \in M$ and $-\sigma \le i < j \le \upsilon$,

$$\limsup_{n\to\infty} -\frac{1}{n} \log \angle(D_x f^n(v_i), D_x f^n(v_j)) \le 0$$

for $v_i \in \mathbf{E}_i^x$ and $v_j \in \mathbf{E}_j^x$.

We refer to the numbers $\lambda_{-\sigma}, \ldots, \lambda_{-1}, \lambda_0, \lambda_1, \ldots, \lambda_\upsilon$ as Lyapunov exponents. For our purposes, it will often be sufficient to deal with a coarser invariant splitting of $T_x(M)$ into stable, center, and unstable subspaces at μ-a.e. $x \in M$, which we shall denote by $\mathbf{E}_x^s \oplus \mathbf{E}_x^c \oplus \mathbf{E}_x^u$. We define $s \equiv \dim(\mathbf{E}^s) = d_{-\sigma} + \cdots + d_{-1}$, $c \equiv \dim(\mathbf{E}^c) = d_0$, and $u \equiv \dim(\mathbf{E}^u) = d_1 + \cdots + d_\upsilon$. We further define $\lambda_{\min} \equiv \min\{|\lambda_{-1}|, \lambda_1\}$.

At μ-a.e. $x \in M$, we are able to define a new inner product and norm adapted to reflect the long-term behavior of the mapping along an orbit. (Excellent references for more details on the following material are [**2**] and [**6**].) We first define the adapted inner product as follows:

$$\langle u, v\rangle_x' = \sqrt{2}\sum_{n=0}^{\infty} \frac{\langle D_x f^{-n}(u), D_x f^{-n}(v)\rangle}{e^{-2n(\lambda_1-\epsilon)}} \quad \text{for } u, v \in \mathbf{E}_x^u,$$

$$\langle u, v\rangle_x' = \sqrt{2}\sum_{n=-\infty}^{\infty} \frac{\langle D_x f^{n}(u), D_x f^{n}(v)\rangle}{e^{2|n|}} \quad \text{for } u, v \in \mathbf{E}_x^c,$$

$$\langle u, v\rangle_x' = \sqrt{2}\sum_{n=0}^{\infty} \frac{\langle D_x f^{n}(u), D_x f^{n}(v)\rangle}{e^{2n(\lambda_{-1}-\epsilon)}} \quad \text{for } u, v \in \mathbf{E}_x^s.$$

Then extend $\langle \cdot, \cdot\rangle'$ to all of $T_x(M)$ by imposing mutual orthogonality with respect to $\langle \cdot, \cdot\rangle'$ on the subspaces $\mathbf{E}_x^u$, $\mathbf{E}_x^c$, and $\mathbf{E}_x^s$. The definition of the adapted norm $\|\cdot\|_x'$ follows as usual from that of the adapted inner product. The factor $\sqrt{2}$ is simply a convenient geometrical factor included to guarantee that $\|\cdot\| \le \|\cdot\|'$. It could be omitted here, at the cost of including a factor of $1/\sqrt{2}$ at numerous other places.

From the above definition, we see that there exists a measurable function $H(x)$ such that $\|\cdot\| \le \|\cdot\|' \le H(x)\|\cdot\|$ and $e^{-\epsilon} < H(f(x))/H(x) < e^{\epsilon}$. Then we can find compact sets $\Lambda_k \subset M$, $k \in \mathbb{N}$, with $\Lambda_k \subset \Lambda_{k+1} \subset \Lambda$ and $\mu\big(\bigcup_{k\in\mathbb{N}} \Lambda_k\big) = 1$, such that for $n \in \mathbb{Z}$ there exists an adapted norm defined on $T(f^n(\Lambda_k))$, denoted by $\|\cdot\|_{k,n}$, which satisfies

$$\|\cdot\|_{\mathrm{R}} \le \|\cdot\|_{k,n} \le \|\cdot\|_{\mathrm{R}} \cdot e^{(k+|n|)\epsilon}, \tag{1}$$

where $\|\cdot\|_{\mathrm{R}}$ refers to the given Riemannian norm on TM. By imposing this norm on the tangent space of an image disk of the exponential chart from a neighborhood of $x \in f^n(\Lambda_k)$ into $\mathbb{R}^{\dim(M)}$, we generate a corresponding metric on this chart. We will call this metric $\rho_{k,n,x}$, or simply $\rho_{k,n}$. We can assume the above relations hold among metrics, i.e.

$$d \le \rho_{k,n} \le d \cdot e^{(k+|n|)\epsilon}, \tag{2}$$

where d is the Riemannian metric.

It is important to notice the 'good' behavior of these new norms with respect to the mapping, in the sense that

$$\|D_{f^n(x)}f(v)\|_{k,n+1} \geq \|v\|_{k,n} \cdot e^{(\lambda_1-\epsilon)} \quad \text{for } v \in \mathbf{E}^u_{f^n(x)},$$
$$\|v\|_{k,n} \cdot e^{\epsilon} \geq \|D_{f^n(x)}f(v)\|_{k,n+1} \geq \|v\|_{k,n} \cdot e^{-\epsilon} \quad \text{for } v \in \mathbf{E}^c_{f^n(x)},$$
$$\|D_{f^n(x)}f(v)\|_{k,n+1} \leq \|v\|_{k,n} \cdot e^{(\lambda_{-1}+\epsilon)} \quad \text{for } v \in \mathbf{E}^s_{f^n(x)}.$$

Note also that, by (1), we may assume that $f^n(\Lambda_k) \subset \Lambda_{k+|n|}$ for $n \in \mathbb{Z}$.

Standard techniques of non-uniform, partial hyperbolic theory (see [**6**] for details) allow us to conclude that:

MAN-1 For any k and $n \in \mathbb{Z}$ and μ-a.e. $x \in \Lambda_k$, there exist local unstable manifolds $W^u_{\text{loc}}(f^n x, e^{-(k+|n|)\epsilon}, \rho_{k,n})$.

MAN-2 f is uniformly expanding with respect to adapted norm distances on these local unstable manifolds; i.e., for y, $z \in W^u_{\text{loc}}(f^n x, e^{-(k+n)\epsilon}, \rho_{k,n})$ and $n \in \mathbb{N}$, we have $f^{-1}y$ and $f^{-1}z \in W^u_{\text{loc}}(f^{n-1}x, e^{-(k+n-1)\epsilon}, \rho_{k,n-1})$, and

$$\rho_{k,n}(y,z) > e^{(\lambda-2\epsilon)} \cdot \rho_{k,n-1}(f^{-1}y, f^{-1}z). \tag{3}$$

MAN-3 Similarly, f^{-1} is uniformly contracting with respect to adapted norm distances on these local unstable manifolds.

MAN-4 By Lusin's theorem, given k, there exists $r(k) > 0$ such that whenever $r < r(k)$, the manifolds $W^u_{\text{loc}}(y, e^{-k\epsilon}, \rho_{k,0})$ of points $y \in B(x,r) \cap \Lambda_k$ form a continuous family of embedded disks, where $B(x,r)$ is the ball of radius r about x in M.

We now review some pertinent facts regarding measurable partitions of our manifold M, considered as a measure space (M,μ) with a probability measure μ defined on its Borel sets. An excellent reference for more details is [**7**].

Suppose ξ is a partition of M. We will let ξ_x denote the element of the partition containing $x \in M$, and M/ξ denote the factor space whose points are the elements of ξ. We will say that ξ is a measurable partition if there exists a countable collection of measurable sets $\mathbf{B} \equiv \{B_n : n \in \mathbb{N}\}$ such that for any $n \in \mathbb{N}$ and $\xi_x \in \xi$, either $\xi_x \subset B_n$ or $\xi_x \subset M - B_n$; and $\xi_x = \cap_{B_n \supset \xi_x} B_n$. We say that $\mathbf{B}$ generates ξ.

Given a measurable partition ξ of (M,μ), there exists a probability measure ν_ξ defined on the factor space M/ξ, and a system of conditional probability measures μ_x defined for μ-a.e. $x \in M$ on its corresponding ξ_x with the following properties:

PROB-1 for any μ-measurable set $A \subset M$, $A \cap \xi_x$ is μ_x-measurable for μ-a.e. $x \in \mathbb{N}$;
PROB-2 $\mu_x(A \cap \xi_x)$ is a ν_ξ-measurable function; and finally,
PROB-3 $\mu(A) = \int_{M/\xi} \mu_x(A \cap \xi_x) d\nu_\xi$.

Let ψ be another measurable partition. We say that ψ refines ξ, and write $\psi > \xi$, if for μ-a.e. $x \in M$, $\psi_x \subset \xi_x$.

In [**1**], Ledrappier and Strelcyn prove the existence of a measurable partition of M with very useful properties. We sketch the construction here.

Fix k with $\mu(\Lambda_k) > 0$, and choose $z \in \text{Supp}(\mu|\Lambda_k)$. We let $\alpha(z,r)$ be the partition with members being the individual sets $W^u_{\text{loc}}(y, e^{-k\epsilon}, \rho_{k,0}) \cap B(z,r)$ for $y \in \Lambda_k \cap B(z,r)$

and $r < r(k)$, along with the complement of their union. Recall that $B(z, r) \subset M$ is the open ball of radius r centered at $z \in M$. Recall also the essential defining property of $r(k)$ from (MAN-4): given k, there exists $r(k) > 0$ such that whenever $r < r(k)$ the manifolds $W^u_{\text{loc}}(y, e^{-k\epsilon}, \rho_{k,0})$ of points $y \in B(z, r) \cap \Lambda_k$ form a continuous family of embedded disks. (For more details on essential properties of these manifolds, see [**1**].) Then we define the partition

$$\xi(z, r) \equiv \bigvee_{i=0}^{\infty} f^i(\alpha(z, r)).$$

There exists a neighborhood $I \subset \mathbb{R}^+$ of zero such that for Lebesgue-a.e. $r_z \in I$, $\xi(z, r_z)$ has the following properties:

PART-1 The forward images of the $W^u_{\text{loc}}(y, e^{-k\epsilon}, \rho_{k,0}) \cap B(z, r)$ cover a set of measure one; i.e., $\mu\big(\bigcup_{i=0}^{\infty} f^i\big(\bigcup_{y \in B(z,r_z)\cap\Lambda_k} W^u_{\text{loc}}(y) \cap B(z, r_z)\big)\big) = 1$. This follows from the ergodicity of the invariant measure μ with respect to f, and the fact that $\mu(B(z, r_z) \cap \Lambda_k) > 0$.

PART-2 $\xi(z, r_z)$ refines $f(\xi(z, r_z))$; i.e., $\xi(z, r_z) > f(\xi(z, r_z))$. This follows from the definition of $\xi(z, r_z)$.

PART-3 $\xi(z, r_z)$ is subordinate to the W^u_{glob} foliation; i.e., for μ-a.e. x, $\xi_x \subset W^u_{\text{glob}}(x)$, and contains a neighborhood of x open in the submanifold topology.

PART-4 For μ-a.e y, $\bigcup_{x\in\xi_y(z,r_z)} B^u(x, 1, d) \subset D_y \subset W^u_{\text{glob}}(y)$ for a ball D_y, open in the unstable submanifold topology, of finite volume computed with respect to d. (This follows from the construction of $\xi(z, r_z)$, and PART-1.)

Given a choice of k and z, we will denote the associated partition $\xi(z, r_z)$ as either $\xi^{z,k}$, ξ^z, or simply ξ when confusion is unlikely.

Define $V(x, n, \delta) \equiv \{y \in W^u_{\text{glob}}(x) : d(f^k x, f^k y) < \delta,\ 0 \le k \le n\}$. Fix any k with $\mu(\Lambda_k) > 0$, and any $z \in \text{Supp}(\mu|\Lambda_k)$. Let $\xi^{z,k}$ denote the measurable partition defined above. We refer to $\xi^{z,k}$ simply as ξ, the ball $B(z, r_z)$ as B, and the induced measure ν_ξ as ν. Note that $h_\mu(f, \xi) \equiv H(\xi | \bigvee_{n=1}^{\infty} f^n\xi) = H(\xi | f\xi)$, since ξ refines $f(\xi)$.

Then the equation proved in §9.2 of [**3**] states that

$$\lim_{\delta\to 0} \liminf_{n\to\infty} -\frac{1}{n} \log \mu_x(V(x, n, \delta)) \ge h_\mu(f, \xi) \quad \text{for } \mu\text{-a.e. } x \in M. \tag{4}$$

We may also use the invariance of μ and the a.e.-uniqueness of conditional measures to deduce that the limit in (4),

$$\lim_{\delta\to 0} \liminf_{n\to\infty} -\frac{1}{n} \log \mu_x(V(x, n, \delta)),$$

is independent of the choice of ξ, despite the apparent dependence which enters through the conditional measures, and that this limit is invariant, hence a.e.-constant by ergodicity. As a consequence, we may reasonably define the μ-a.e. constant, ξ-independent entropy along the unstable foliation as being $h^u_\mu \equiv h_\mu(f, \xi)$. Note that (4) then becomes

$$\lim_{\delta\to 0} \liminf_{n\to\infty} -\frac{1}{n} \log \mu_x(V(x, n, \delta)) \ge h^u_\mu \quad \text{for } \mu\text{-a.e. } x \in M.$$

3. *Proofs*

PROPOSITION 3.1. *Let f be a $C^{1+\alpha}$ diffeomorphism of a compact manifold M which preserves an ergodic probability measure μ. Then for μ-a.e. $x \in M$, there exists a disk D_x, open in the submanifold topology of $W^u_{\text{glob}}(x)$, with $x \in D_x \subset W^u_{\text{glob}}(x)$ and $G(D_x) \geq h^u_\mu$.*

Proof. Since $\mu_x(V(x,n,\delta))$ is a decreasing function of δ, we note that for any sequence $\{\delta_n\}_{n=0}^\infty$ of positive numbers with $\lim_{n\to\infty} \delta_n = 0$,

$$\liminf_{n\to\infty} -\frac{1}{n} \log \mu_x(V(x,n,\delta_n)) \geq \liminf_{n\to\infty} -\frac{1}{n} \log \mu_x(V(x,n,\delta))$$

for fixed $\delta > 0$. Then we may conclude that

$$\liminf_{n\to\infty} -\frac{1}{n} \log \mu_x(V(x,n,\delta_n)) \geq h^u_\mu \quad \text{for } \mu\text{-a.e. } x \in M. \tag{5}$$

Let ξ be any partition possessing the properties (PART). Choose $\epsilon > 0$, and generate the associated sets Λ_k, $k \in \mathbb{N}$, as described in the preliminaries. Fix k with $\mu(\Lambda_k) > 0$. For μ-a.e. $x \in \Lambda_k$, $\mu_x(\Lambda_k) > 0$, while for ν-a.e. ξ_x, (5) holds at μ_x-a.e. $y \in \xi_x$, since it holds μ-a.e. on M. Thus, for μ-a.e. point $p \in \Lambda_k$ and any sequence of positive real numbers $\{\delta_n\}_{n=0}^\infty$ with $\lim_{n\to\infty} \delta_n = 0$,

$$\liminf_{n\to\infty} -\frac{1}{n} \log \mu_p(V(x,n,\delta_n)) \geq h^u_\mu \quad \text{for } \mu_p\text{-a.e. } x \in \xi_p \cap \Lambda_k.$$

As a consequence, for $j \in \mathbb{N}$ we may find an increasing sequence of compact sets $\Lambda_{k,j} \subset \xi_p \cap \Lambda_k$ and an increasing sequence of integers $n(j)$ such that

$$\mu_p(\Lambda_{k,j}) > \mu_p(\Lambda_k) \cdot \left(1 - \frac{1}{j}\right) \tag{6}$$

and

$$-\frac{1}{n} \log \mu_p(V(x,n,\delta_n)) > h^u_\mu - \frac{1}{j} \quad \text{for } x \in \Lambda_{k,j} \text{ and } n > n(j).$$

Equivalently,

$$\mu_p(V(x,n,\delta_n)) \leq e^{-n\cdot(h^u_\mu - (1/j))} \quad \text{for } x \in \Lambda_{k,j} \text{ and } n > n(j). \tag{7}$$

Let F_n be a maximum cardinality (n,δ_n)-separated subset of $\Lambda_{k,j}$, and C_n be the cardinality of F_n. Then

$$\Lambda_{k,j} \subset \bigcup_{x\in F_n} V(x,n,\delta_n).$$

Consequently,

$$\mu_p(\Lambda_{k,j}) \leq C_n \cdot \sup_{x\in\Lambda_{k,j}} \mu_p(V(x,n,\delta_n)),$$

so that

$$C_n > \mu_p(\Lambda_k) \cdot \left(1 - \frac{1}{j}\right) \cdot e^{n\cdot(h^u_\mu - (1/j))} \quad \text{for } n > n(j), \tag{8}$$

by (6) and (7). Recall the relations between the Riemannian metric d and the adapted metric $\rho_{k,n}$ stated in (2); i.e.,

$$d \leq \rho_{k,n} \leq d \cdot e^{(k+|n|)\epsilon}.$$

Define $\delta_n \equiv e^{-(k+n)\epsilon}$ for $n \in \mathbb{N}$. For any distinct x and y in F_n, we have $d(f^i x, f^i y) > \delta_n$ for some $i \leq n$, and thus $\rho_{k,i}(f^i x, f^i y) > \delta_n$. Since f uniformly expands balls about x and y of radius δ_m with respect to the adapted metric $\rho_{k,m}$ for all $m \geq 0$ by (MAN-2), we have $\rho_{k,l}(f^l x, f^l y) \geq \delta_n$ for $i \leq l \leq n$. Thus, at time n, all $f^n x$ for $x \in F_n$ are mutually separated by an adapted distance of at least δ_n, and thus by a Riemannian distance of at least $\delta_n \cdot e^{-(k+n)\epsilon} = \delta_n^2$ by (2) and the definition of δ_n. Hence their $\frac{1}{2} \cdot \delta_n^2$ balls in $W^u_{\text{glob}}(p)$ are mutually disjoint. Note that the volume of a $\frac{1}{2} \cdot \delta_n^2$ ball in $W^u_{\text{glob}}(p)$ is $\geq G \cdot \delta_n^{2u}$, where u is the dimension of the unstable manifolds and G is a constant depending only on the geometry of the manifold. Then

$$\sum_{x \in F_n} V(B(f^n x, \tfrac{1}{2}\delta_n^2)) \geq C_n \cdot G \cdot \delta_n^{2u}, \tag{9}$$

and so by (8)

$$\sum_{x \in F_n} V(B(f^n x, \tfrac{1}{2}\delta_n^2)) \geq \text{constant} \cdot e^{nh^u_\mu} \cdot e^{-((1/j)+2u\epsilon)n} \quad \text{for } n > n(j).$$

From (PART-4), we know that there exists a disk $D_p \subset W^u_{\text{glob}}(p)$ of volume $V(D_p)$ with $\bigcup_{x \in \Lambda_k \cap \xi_p} B(x, 1) \subset D_p$. (MAN-3) allows us to conclude that $f^{-n}(B(f^n x, \frac{1}{2}\delta_n^2)) \subset D_p$ for any $x \in \Lambda_k \cap \xi_p$.

Consequently, whenever $n > n(j)$, we know that

$$\frac{1}{n} \log \frac{V(f^n D_p)}{V(D_p)} > \frac{\text{constant}}{n} + h^u_\mu - \left(\frac{1}{j} + 2u\epsilon\right),$$

and thus

$$\liminf_{n \to \infty} \frac{1}{n} \log \frac{V(f^n D_p)}{V(D_p)} \geq h^u_\mu - 2u\epsilon. \tag{10}$$

This holds for μ-a.e. $p \in \Lambda_k$, and since $\mu(\bigcup_{k=1}^\infty \Lambda_k) = 1$, (10) holds for μ-a.e. $p \in M$. By considering a sequence of values of ϵ which decrease to zero, we will show that for μ-a.e. $p \in M$,

$$\liminf_{n \to \infty} \frac{1}{n} \log \frac{V(f^n D_p)}{V(D_p)} \geq h^u_\mu. \tag{11}$$

To this end, choose $\epsilon' < \epsilon$, and repeat the above argument verbatim, with every quantity having its primed analog. (Note, however, that ξ was chosen before ϵ, and remains fixed throughout this proof.) For all p in some set G' with $\mu(G') = 1$, we know that D_p satisfies

$$\liminf_{n \to \infty} \frac{1}{n} \log \frac{V(f^n D_p)}{V(D_p)} \geq h^u_\mu - 2u\epsilon', \tag{12}$$

while for all p in some set G with $\mu(G) = 1$, we know that D_p satisfies (10). Then for all $p \in G \cap G'$, for which $\mu(G \cap G') = 1$, D_p will satisfy (12). Repeating this argument countably many times with a sequence of ϵ' values which decrease to zero will then give us (11), completing the proof of the proposition. □

Proof of Theorem 1.1. If f is C^{1+1}, then $h^u_\mu = h_\mu$ by equation (5.3) in [**2**]. Then the theorem follows from the preceding proposition. □

Proof of Corollary 1.1. By the variational principle, the supremum over ergodic, invariant probability measures μ of h_μ is h. Corollary 1.1 then follows from the theorem. □

Proof of Corollary 1.2. If μ is a measure of maximal entropy, our theorem holds with $h_\mu = h \equiv$ topological entropy, by the variational principal. In [**10**], Yomdin shows that for any C^∞ map f and disk D,

$$h \geq \limsup_{n\to\infty} \frac{1}{n} \log^+ \frac{V(f^n(D))}{V(D)}.$$

Since our unstable manifolds are C^∞, the second statement in the corollary follows immediately. In [**5**], Newhouse shows that, if f is C^∞, a measure μ of maximal entropy exists. Since Supp(μ) is non-empty, at least one such disk D with $\Gamma(D) = h$ exists. □

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