A Multiparameter, Zero Density Subsequence Ergodic Theorem.

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Abstract. We generalize a result of L. Sucheston on obtaining multiparameter ergodic theorems from their single parameter versions. This result is then employed to prove a multiparameter, subsequence ergodic theorem for operator averages along special zero density subsequences.

1 Introduction

In [9], L. Sucheston presents a method for obtaining multiparameter ergodic theorems from their single parameter versions. In the second section of this article, we present a generalization of this method.

In [2] and [10], J. Bourgain and M. Weirdl establish the a.e. convergence of averages of the form $\frac{1}{k}\sum_{i=1}^{k} f \circ \tau^{n_i}$, where τ is an automorphism of a finite measure space (X, F, μ), $f \in L_p(X, F, \mu)$, $1 < p < \infty$, and $\mathbf{n} = (n_i)_{i=1}^{\infty}$ is either the sequence of primes, or a polynomial sequence, the latter being defined as the sequence of values $(\phi(n))_{n=1}^{\infty}$ assumed by a polynomial ϕ of degree ≥ 2, having integer coefficients. We call such sequences *B-sequences*. If we define the density of a sequence $\mathbf{n}$ as $d(\mathbf{n}) = \lim_N \frac{|\mathbf{n}\cap[1,N]|}{N}$, we see that B-sequences have zero density.

In the third section of this article, we note that the work of R. Jones, J. Olsen, and Weirdl allows us to move the results of Bourgain and Weirdl to more general, operator theoretic settings. In conjunction with the results of section two, this will allow us to obtain a multiparameter, operator theoretic ergodic theorem for averages taken along B-sequences.

2 Obtaining multiparameter theorems

Let $(\mathbf{B}, \|\cdot\|_{\mathbf{b}}, \preceq)$, $(\mathbf{C}, \|\cdot\|_{\mathbf{c}}, \preceq)$, and $(\mathbf{D}, \|\cdot\|_{\mathbf{d}}, \preceq)$ be Banach lattices of functions over a common measure space (X, F, μ), so that convergence in order corresponds to a.e. convergence. Suppose that $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ have σ-order continuous norms. (Recall that we say $(\mathbf{B}, \|\cdot\|_{\mathbf{b}}, \preceq)$ has σ-order continuous norm if $\|b_n\|_b \downarrow 0$ whenever $b_n \downarrow \theta$. Examples of Banach lattices with σ-order continuous norms are $L_p(X, F, \mu)$, $1 \leq p < \infty$). If $T\colon \mathbf{B} \to \mathbf{C}$

and $S\colon \mathbf{B} \to \mathbf{C}$, we say S *dominates* T if $|Tb| \preceq S|b|\ \forall b \in \mathbf{B}$, and say that T is *positively dominated*.

Let $\{D, \preceq\}$ be a countable directed set. Let $\{(d,n) \in D \times \mathbf{N}, \preceq'\}$ be the countable directed set with order defined by $(d_1, n_1) \preceq' (d_2, n_2)$ iff $d_1 \preceq d_2$ and $n_1 \leq n_2$.

Let $\{U_d, d \in D\}$ be a net of continuous, linear operators $U_d\colon \mathbf{C} \to \mathbf{D}$ with $\lim_d U_d c = U_\infty$ a.e. $\forall c \in \mathbf{C}$. Let $\{V_d, d \in D\}$ be a net of positive, linear (hence continuous) operators $V_d\colon \mathbf{C} \to \mathbf{D}$, with each V_d dominating the corresponding U_d. Define the maximal operator associated with the net (V_d) as $M_V c = \sup_d |V_d c|$, and suppose that $M_V\colon \mathbf{C} \to \mathbf{D}$. Note that it is sufficient, but not necessary, to have $\|M_V c\|_d \leq K_c \cdot \|c\|_c\ \forall c \in \mathbf{C}$ and some $K_c < \infty$.

Let $\{T_n, n \in \mathbf{N}\}$ be a sequence of continuous, linear operators $T_n\colon \mathbf{B} \to \mathbf{C}$ with $\lim_n T_n b = T_\infty b$ a.e. $\forall b \in \mathbf{B}$. Let $\{S_n,\ n \in \mathbf{N}\}$ be a sequence of positive, linear (hence continuous) operators $S_n\colon \mathbf{B} \to \mathbf{C}$; with each S_n dominating the corresponding T_n, and $M_S\colon \mathbf{B} \to \mathbf{C}$ (where $M_S b = \sup_n |S_n b|$).

Lemma 1. Given the above hypothesis, M_V has the properties:

- i) M_V is a positive map from $\mathbf{C}$ into $\mathbf{D}$;
- ii) for $\alpha \geq 0$ and $f \succeq \theta$, $M_V(\alpha f) = \alpha \cdot (M_V f)$;
- iii) for $f \succeq \theta$ and $g \succeq \theta$, $M_V(f+g) \preceq M_V f + M_V g$, and;
- iv) M_V is continuous at θ, with $M_V \theta = \theta$.

Proof. i), ii), and iii) are clear. iv) Each V_d is linear, so $M_V \theta = \theta$. Suppose M_V is not continuous at θ. Then $\exists (f_n)_{n=1}^\infty \subset \mathbf{C}$ such that $\|\,|f_n|\,\|_c < \frac{1}{n^3}\ \forall n \in \mathbf{N}$, while $\|M_V f_{n_k}\|_d > \epsilon$ for some $\epsilon > 0$ and all n_k in some sequence $(n_k)_{k=1}^\infty$. Since M_V is positive, $\|M_V |f_{n_k}|\,\|_d > \|M_V f_{n_k}\|_d > \epsilon\ \forall n_k$. Thus, we may take the functions f_n to be non-negative. By i), $\|n \cdot f_n\|_c < \frac{1}{n^2}$ and $\|M_V(n_k \cdot f_{n_k})\|_d = \|n_k \cdot (M_V f_{n_k})\|_d = n_k \cdot \|M_V f_{n_K}\|_d > n_k \cdot \epsilon\ \forall n_k$. Since $\|\sum_{n=1}^\infty n \cdot f_n\|_c \leq \sum_{n=1}^\infty \|n \cdot f_n\|_c \leq \sum_{n=1}^\infty \frac{1}{n^2} < \infty$, $f = \sum_{n=1}^\infty n \cdot f_n$ exists in $\mathbf{C}$. Each f_n is non-negative, so $f \succeq n_k \cdot f_{n_k}\ \forall n_k$. M_V is positive, so $n_k \cdot \epsilon \leq \|M_V(n_k \cdot f_{n_k})\|_d \leq \|M_V f\|_d\ \forall k$. This contradicts the fact that $M_V\colon \mathbf{C} \to \mathbf{D}$. Thus, M_V is continuous at θ.

Corollary 2. M_S is a special case of M_V, and thus has properties i)–iv).

Theorem 3. Given the above hypotheses, we have:

- i) $\lim_{(d,n)} U_d T_n f = U_\infty T_\infty f\ \forall f \in \mathbf{B}$, and;

- ii) $M_{VS}\colon \mathbf{B} \to \mathbf{D}$, where $M_{VS}f = \sup_{(d,n)} |V_d S_n f|$.

Proof. i) $\overline{\lim}_{(d,n)}|U_dT_nf - U_\infty T_\infty f| \leq \overline{\lim}_{(d,n)}|U_dT_nf - U_dT_\infty f| + \overline{\lim}_{(d,n)}|U_dT_\infty f - U_\infty T_\infty f| = \overline{\lim}_{(d,n)}|U_d(T_nf - T_\infty f)| + \theta \preceq \overline{\lim}_{(d,n)}V_d|T_nf - T_\infty f|$.

For each $k \in \mathbf{N}$, let $g_k = \sup_{n\geq k}|T_nf - T_\infty f| \preceq 2\cdot M_S|f| \in \mathbf{C}$. Note that $g_k \downarrow \theta$ and $\|g_k\|_c \downarrow 0$. For each fixed k, $\overline{\lim}_{(d,n)}V_d|T_nf - T_\infty f| \preceq \overline{\lim}_{(d,n)}V_dg_k$, since each V_d is positive. Then $\overline{\lim}_{(d,n)}|U_dT_nf - U_\infty T_\infty f| \preceq \overline{\lim}_{(d,n)}V_dg_k \leq M_Vg_k\ \forall k$. Now let $k \to \infty$. M_V is positive, and $g_k \downarrow \theta$, so $M_Vg_k \downarrow g$ for some $g \succeq \theta$. By the lemma, $\|M_Vg_k\|_d \downarrow 0$. Thus, $g = \theta$ a.e., and we have $\overline{\lim}_{(d,n)}|U_dT_nf - U_\infty T_\infty f| = \theta \Rightarrow \lim_{(d,n)} U_dT_nf = U_\infty T_\infty f$ a.e. $\forall f \in \mathbf{B}$.

ii) For any $f \in \mathbf{B}$, we have $M_{VS}f = \sup_{(d,n)}|V_dS_nf| \leq \sup_{(d,n)} V_d|S_nf| \leq \sup_d V_d(\sup_n |S_nf|) = \sup_d V_d(M_Sf) = M_V(M_Sf) \in \mathbf{D}$. Thus, $M_{VS}f \in \mathbf{D}$.

Theorem 3 will allow us to inductively obtain multiparameter theorems from their single parameter counterparts. Sucheston presented this result in [9] for positive operators, and thus for positively dominated operators, provided that we have a.e. convergence for the dominating operators. Here, we have removed the latter requirement. In [8], Olsen noted that the result may be used to prove multiparameter weighted theorems, and thus multiparameter subsequence theorems along sequences with non-zero density. Here, we deal with zero density B-sequences.

3 Result

Fix any p, $1 < p < \infty$. Let (X, F, μ) be a finite measure space, $\tau\colon X \to X$ be an automorphism, and $T\colon L_p(X,F,\mu) \to L_p(X,F,\mu)$. We say $T_\tau\colon L_p \to L_p$, $T_\tau f = f \circ \tau$ is the operator on L_p induced by τ. If $\mathbf{n} = (n_i)_{i=1}^{\infty}$ is an increasing sequence of positive integers, we define $A_k(\mathbf{n}, T)f = \frac{1}{k}\cdot\sum_{i=1}^{k} T^{n_i}f$ and $M(\mathbf{n}, T)f = \sup_k A_kf$ for $f \in L_p$.

We say the operator T:

i) admits a dominated estimate in L_p along $\mathbf{n}$ with constant C if $\|M(\mathbf{n}, T)f\|_p \leq C \cdot \|f\|_p\ \forall f \in L_p$ and some $C < \infty$.

ii) is power bounded with power bound K if $\|T^nf\|_p \leq K \cdot \|f\|_p\ \forall n \in \mathbf{N}$, $f \in L_p$, and some $K < \infty$.

iii) is a quasi-isometry if $0 < C_1\|f\|_p^p \leq \frac{1}{L_n}\sum_{k=0}^{L_n-1}\|T^kf\|_p^p \leq C_2\|f\|_p^p < \infty$ for some C_1 and C_2, and some increasing sequence of positive integers $\mathbf{L} = (L_n)_{n=1}^{\infty}$.

iv) is a Lamperti operator if it maps functions with disjoint supports to functions with disjoint supports.

v) is a Dunford-Schwartz operator if $\|T\|_1 \leq 1$ and $\|T\|_\infty \leq 1$.

As mentioned in the introduction, if $\mathbf{n}$ is a B-seqeunce, Bourgain and Weirdl have established the a.e. convergence of $A_k(\mathbf{n}, T_\tau)f$ $\forall f \in L_p$. They have also shown that T_τ admits a dominated estimate in L_p along $\mathbf{n}$.

In [5], R. Jones and J. Olsen show that if there exists an aperiodic, positive invertible isometry $S\colon L_p \to L_p$ that admits a dominated estimate in L_p along $\mathbf{n}$ with constant C; then any $T\colon L_p \to L_p$ which is dominated by a positive contraction also admits a dominated estimate in L_p along $\mathbf{n}$ with constant C, and if $T\colon L_p \to L_p$ is a power bounded Lamperti operator with power bound K, then T admits a dominated estimate in L_p along $\mathbf{n}$ with constant $K \cdot C$. (The former builds on the work of Akcoglu in [1] and Ionescu-Tulcea in [4]; the latter on that of C.-H. Kan in [7].) Recall that a Dunford-Schwartz operator is dominated by its linear modulus, which is a positive L_p contraction; and that a Lamperti operator T is dominated by another Lamperti operator S satisfying $|Sf| = T|f|$ $\forall f \in L_p$. (S is once again the linear modulus of T.) Taking S to be the aperiodic, positive, invertible isometry induced by any aperiodic automorphism τ, we see that positive contractions, Dunford-Schwartz operators, and power-bounded Lamperti operators admit dominated estimates along B-sequences. This is relevant to us, in that it means that $M(\mathbf{n}, T)$ maps L_p to L_p whenever $\mathbf{n}$ is a B-sequence and T is either a positive contraction, a Dunford-Schwartz operator, or a power-bounded Lamperti operator.

In [5], Jones and Olsen also show that if we have a.e. convergence of $A_k(\mathbf{n}, T_\tau)f$ $\forall f \in L_p$ for some automorphism τ, then we have a.e. convergence of $A_k(\mathbf{n}, T)f$ $\forall$ bounded $f \in L_p$ when T is a Dunford-Schwartz operator. Bounded functions constitute a dense subset in L_p. Thus, by Banach's principle, we have a.e. convergence of $A_k(\mathbf{n}, T)f$ $\forall f \in L_p$ when T is a Dunford-Schwartz operator and $\mathbf{n}$ is a B-sequence.

In [6], Jones, Olsen, and Weirdl establish a.e. convergence of $A_k(\mathbf{n}, T)f$ $\forall f \in L_p$ whenever $\mathbf{n}$ is a B-sequence and T is either a positive contraction, or a Lamperti quasi-isometry.

Let us call an operator *good* if it is a positive contraction, a Dunford-Schwartz operator, or a power bounded, Lamperti quasi-isometry. Then, for $1 < p < \infty$, $\mathbf{n}$ a B-sequence, and T a good operator, we have a.e. convergence of $A_k(\mathbf{n}, T)f$ $\forall f \in L_p$, and $M(\mathbf{n}, T)\colon L_p \to L_p$. Further, each good operator is dominated by a positive operator which admits a dominated estimate along

B-sequences.

We are now in a position to employ Theorem 3 to prove our principal result, which we state as Theorem 4.

Theorem 4. Let (X, F, μ) be a finite measure space, and $L_p = L_p(X, F, \mu)$ for some fixed p, $1 < p < \infty$. Choose $m \in \mathbf{N}$. For $1 \leq i \leq m$, let $T_i : L_p \to L_p$ be a good operator; S_i be a positive operator which dominates T_i, and which admits a dominated estimate along B-sequences; $\mathbf{n}_i = (n_{i,k_i})_{k_i=1}^{\infty}$ be a B-sequence; and define $\lim_{k_i} A_{k_i}(\mathbf{n}_i, T_i)f = A_{i\infty}f$. Define $M_m f = \sup_{k_1,\ldots,k_m} |A_{k_1}(\mathbf{n}_1, S_1) \cdots A_{k_m}(\mathbf{n}_m, S_m)f|$.

Then $M_m : L_p \to L_p$ and $\lim_{k_1} \cdots \lim_{k_m} A_{k_1}(\mathbf{n}_1, T_1) \cdots A_{k_m}(\mathbf{n}_m, T_m)f = A_{1\infty} \cdots A_{m\infty} f \forall f \in L_p$, the limit being taken as $k_1, \ldots, k_m \to \infty$ independently.

Proof. We proceed by induction. From our previous discussion, the theorem holds for $m = 1$. Suppose it is true for r operators. Then, in Theorem 3, let D be $\{\mathbf{N}^{\mathbf{r}}, \preceq\}$, with $(s_1, \ldots, s_r) \preceq (t_1, \ldots, t_r)$ iff $s_1 \leq t_1, \ldots, s_r \leq t_r$. Let $\mathbf{B} = \mathbf{C} = \mathbf{D} = \mathbf{L_p}$. Let $\{U_d, d \in D\}$ be the net of operators of the form $A_{k_1}(\mathbf{n}_1, T_1) \cdots A_{k_r}(\mathbf{n}_r, T_r)$, with $(k_1, \ldots, k_r) \in D$. Let $\{V_d,\ d \in D\}$ be the net of operators of the form $A_{k_1}(\mathbf{n}_1, S_1) \cdots A_{k_r}(\mathbf{n}_r, S_r)$, with $(k_1, \ldots, k_r) \in D$. Let T_n be $A_n(\mathbf{n}_{r+1}, T_{r+1})$, and S_n be $A_n(\mathbf{n}_{r+1}, S_{r+1})$. By the induction hypothesis, we have $\lim_d U_d f = A_{1\infty} \cdots A_{r\infty} f\ \forall f \in L_p$, and $M_V : L_p \to L_p$; and we also have $\lim_n T_n f = A_{r+1,\infty} f\ \forall f \in L_p$, and $M_S : L_p \to L_p$. Then, by Theorem 3, we have our result for $r + 1$ operators, and thus for m operators.

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